9.6 FAMILIES OF RIGHT TRIANGLES

Objectives

After studying this section, you will be able to

- Recognize groups of whole numbers known as Pythagorean triples
- Apply the Principle of the Reduced Triangle

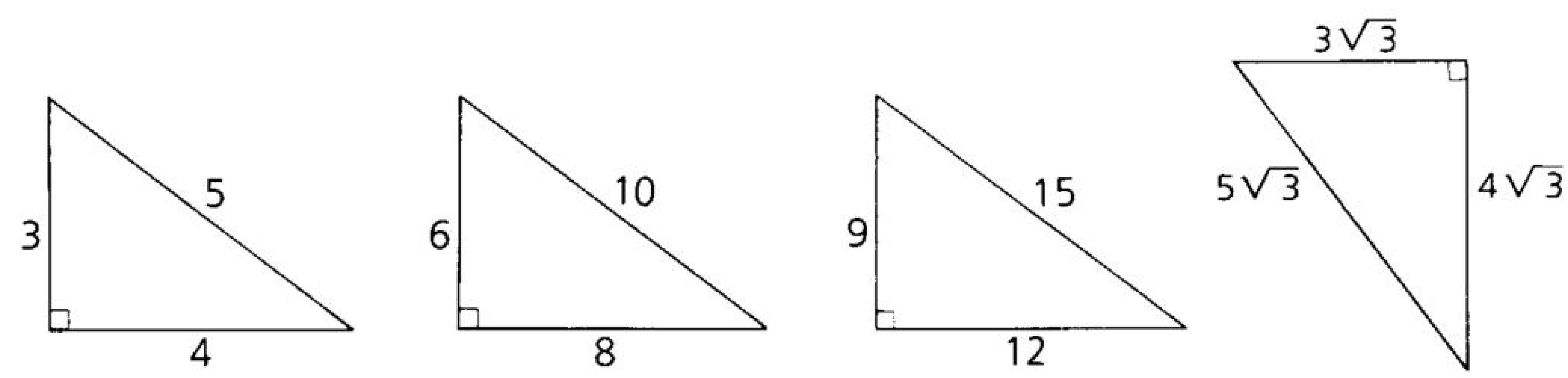
Part One: Introduction

Pythagorean Triples

In this section we consider some combinations of whole numbers that satisfy the Pythagorean Theorem. Knowing these combinations is not essential, but knowing some of them can save you appreciable time and effort.

Definition Any three whole numbers that satisfy the equation $a^2 + b^2 = c^2$ form a ***Pythagorean triple.***

Below is a set of right triangles you have encountered many times in this chapter. Do you see how the triangles are related?



These four triangles are all members of the (3, 4, 5) family. For example, the triple (6, 8, 10) is $(3 \cdot 2, 4 \cdot 2, 5 \cdot 2)$.

Even though the last triangle, $(3\sqrt{3}, 4\sqrt{3}, 5\sqrt{3})$, is a member of the (3, 4, 5) family, the measures of its sides are not a Pythagorean triple because they are not whole numbers.

Other common families are

(5, 12, 13), of which (15, 36, 39) is another member

(7, 24, 25), of which (14, 48, 50) is another member

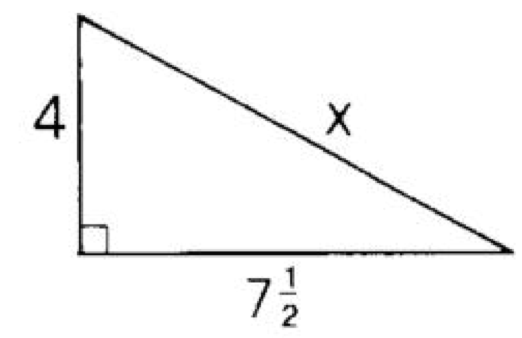
(8, 15, 17), of which $\left(4, 7\frac{1}{2}, 8\frac{1}{2}\right)$ is another member

There are infinitely many families, including (9, 40, 41), (11, 60, 61), (20, 21, 29), and (12, 35, 37), but most are not used very often.

The Principle of the Reduced Triangle

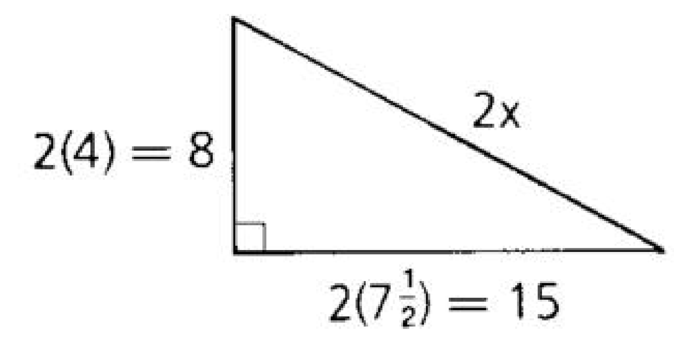
The following problem shows how a knowledge of Pythagorean triples can be useful even in situations where their applicability is not immediately apparent.

Example 1 *Given: The right triangle shown*
Find: x



The fraction may complicate our work, and we may not wish to complete a long calculation to solve $4^2 + \left(7\frac{1}{2}\right)^2 = x^2$.

An alternative is to find a more easily recognized member of the same family. We multiply each side by the denominator of the fraction, 2. Clearly, the family is (8, 15, 17). Thus, $2x = 17$ and $x = 8\frac{1}{2}$ (in the original triangle).

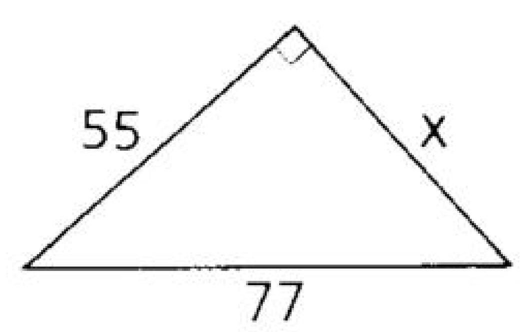


Principle of the Reduced Triangle

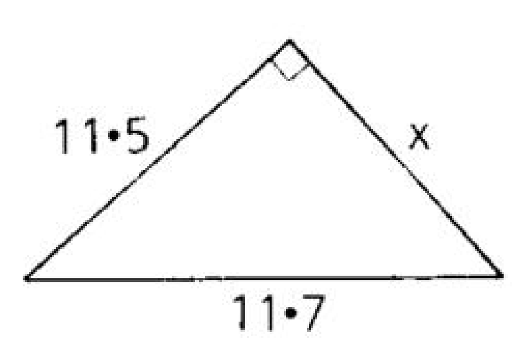
1. Reduce the difficulty of the problem by multiplying or dividing the three lengths by the same number to obtain a similar, but simpler, triangle in the same family.
2. Solve for the missing side of this easier triangle.
3. Convert back to the original problem.

The next example shows that the method may save time even if the sides of the "reduced" triangle are not a proper Pythagorean triple.

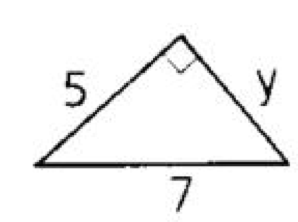
Example 2 *Find the value of* x.



First, notice that both 55 and 77 are multiples of 11. Then reduce the problem to an easier problem as shown below.



is in the family

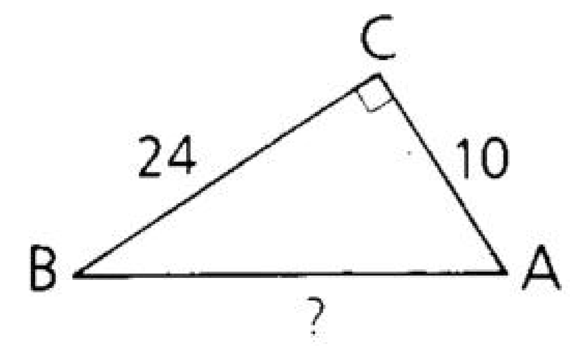


$$\begin{aligned} \text{where } 5^2 + y^2 &= 7^2 \\ 25 + y^2 &= 49 \\ y^2 &= 24 \\ y^2 &= \pm 2\sqrt{6} \quad \text{(Reject } -2\sqrt{6}.\text{)} \end{aligned}$$

Thus, $x = 11 \cdot 2\sqrt{6} = 22\sqrt{6}$.

Part Two: Sample Problems

Problem 1 *Find AB.*



Solution

Method One:

(10, 24, ?) belongs to the (5, 12, 13) family.

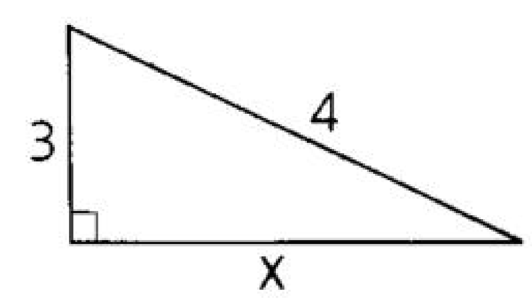
$$10 = 5 \cdot 2$$
$$24 = 12 \cdot 2$$

So $AB = 13 \cdot 2 = 26$

Method Two:

$$10^2 + 24^2 = (AB)^2$$
$$100 + 576 = (AB)^2$$
$$676 = (AB)^2$$
$$\pm\sqrt{676} = AB \quad \text{(Reject } -\sqrt{676}.\text{)}$$
$$26 = AB$$

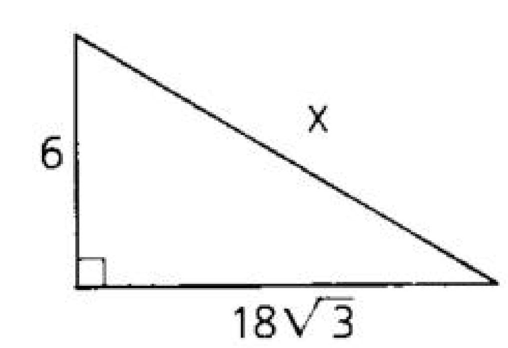
Problem 2 *Find x.*

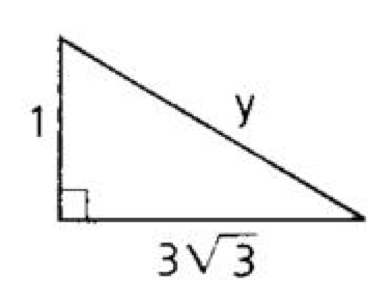


Solution You may think that 5 is the answer, but in a (3, 4, 5) triangle the 5 must represent the length of the hypotenuse. Therefore, we are stuck with the long way.

$$3^2 + x^2 = 4^2$$
$$x^2 = 7$$
$$x = \pm\sqrt{7} \quad \text{(Reject } -\sqrt{7}.\text{)}$$
$$x = \sqrt{7}$$

Problem 3 *Find the hypotenuse of the right triangle.*





Solution

Method One:
Reduced-Triangle Principle

Divide each given length by 6 to obtain the reduced similar triangle.

$$1^2 + \left(3\sqrt{3}\right)^2 = y^2$$
$$1 + 27 = y^2$$
$$\pm\sqrt{28} = y$$
$$\pm 2\sqrt{7} = y \quad \text{(Reject } -2\sqrt{7}.\text{)}$$

Now multiply by 6 to convert back to the original triangle.

$$x = 6\left(2\sqrt{7}\right) = 12\sqrt{7}$$

Method Two:
Pythagorean Theorem

$$6^2 + \left(18\sqrt{3}\right)^2 = x^2$$
$$36 + 972 = x^2$$
$$1008 = x^2$$
$$\sqrt{1008} = x$$
$$\pm\sqrt{144 \cdot 7} = x$$

(Would you have discovered those factors?)

Reject the negative root.

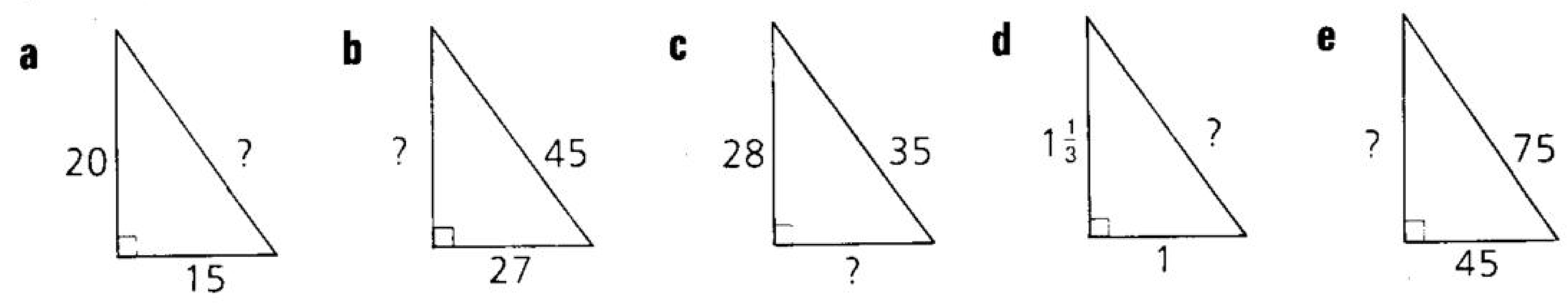
$$12\sqrt{7} = x$$

Part Three: Problem Sets

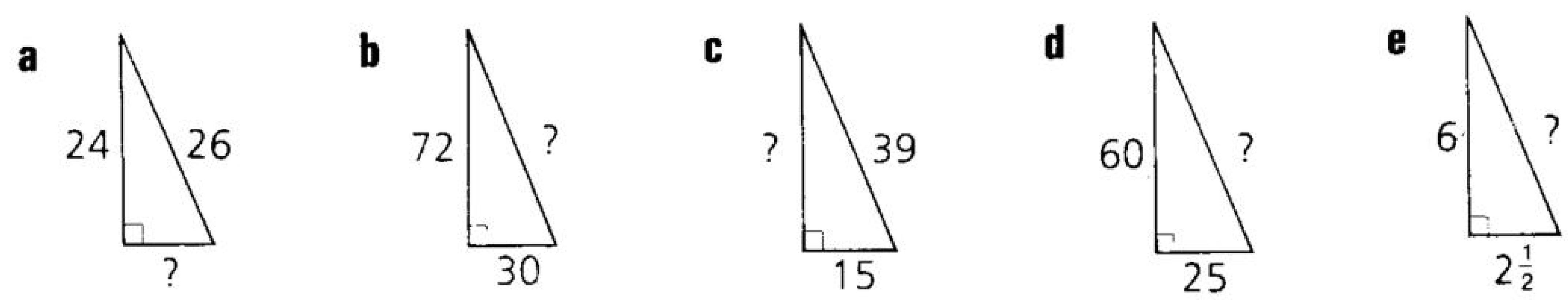
Problem Set A

In problems 1–5, find the missing side in each triangle.

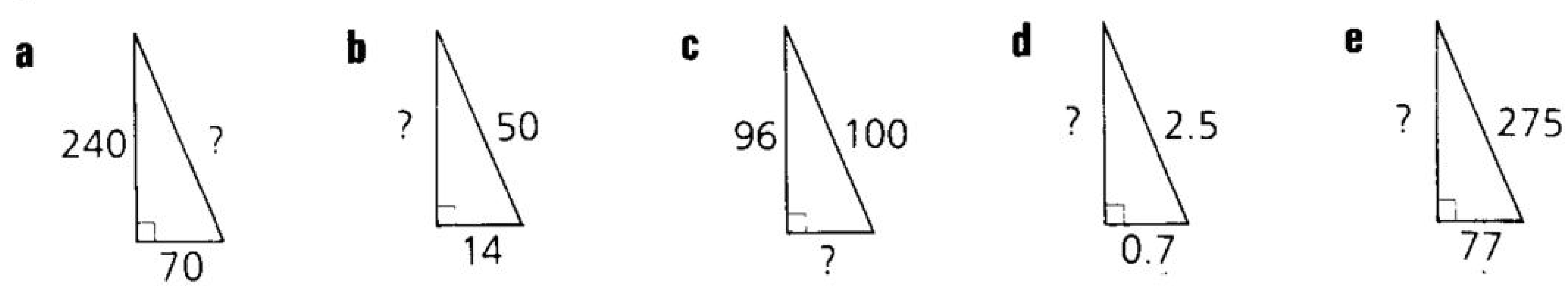
1 (3, 4, 5)



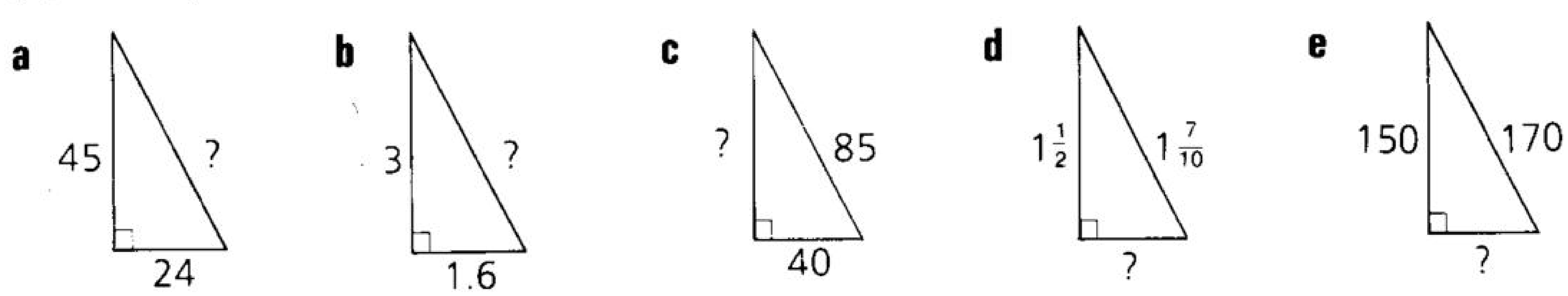
2 (5, 12, 13)



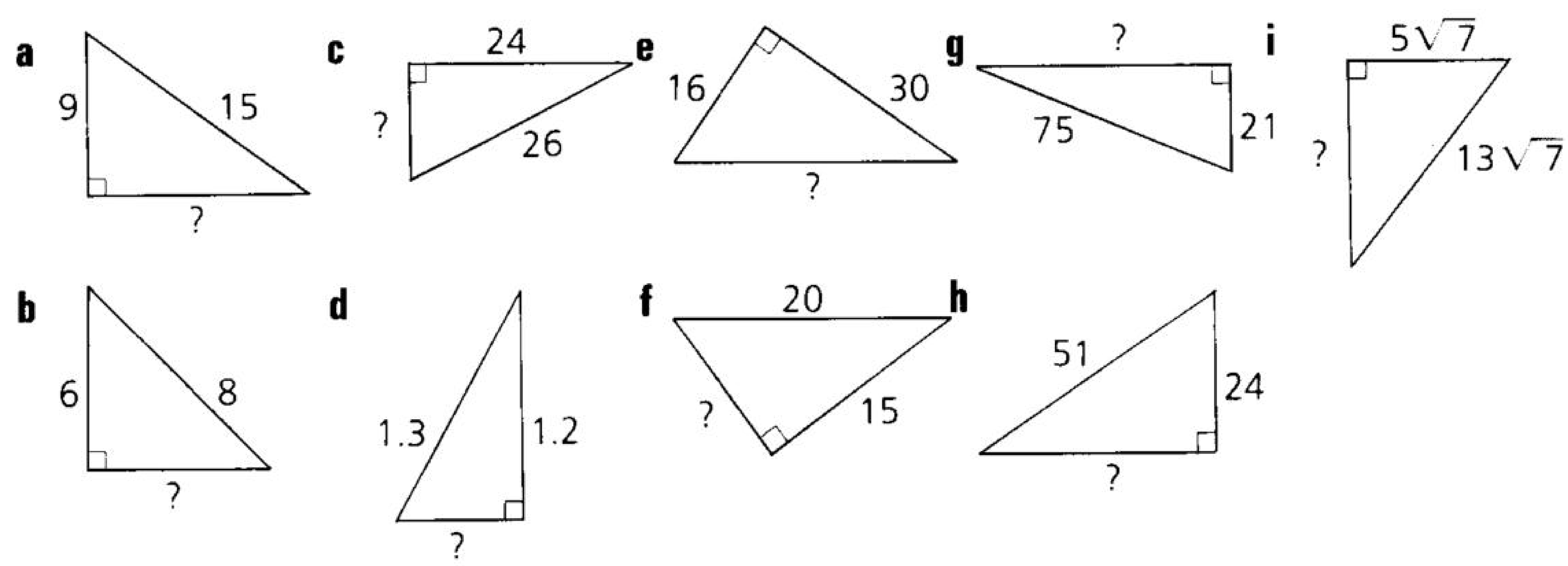
3 (7, 24, 25)



4 (8, 15, 17)



5 Mixed

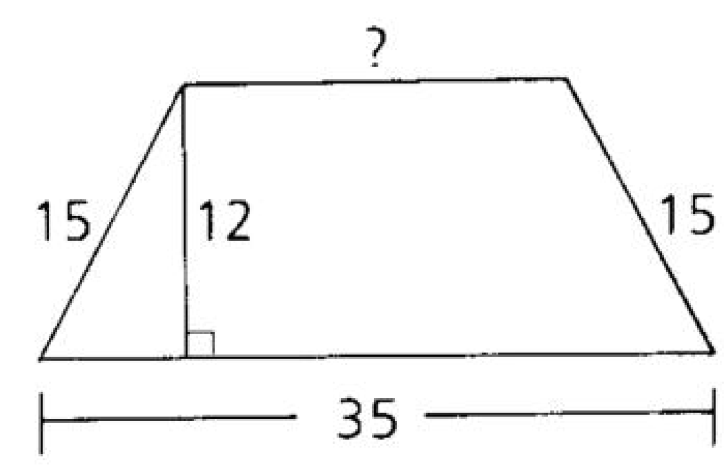


Problem Set A, *continued*

6 Find the diagonal of a rectangle whose sides are 20 and 48.

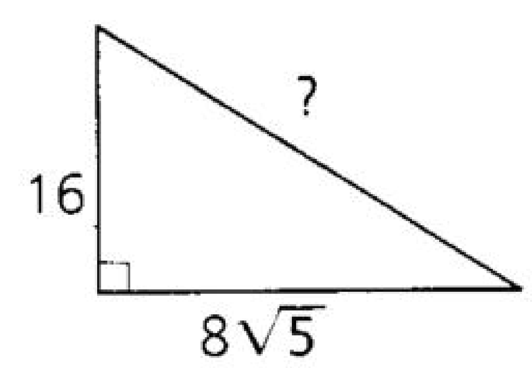
7 Find the perimeter of an isosceles triangle whose base is 16 dm and whose height is 15 dm.

8 Find the length of the upper base of the isosceles trapezoid.

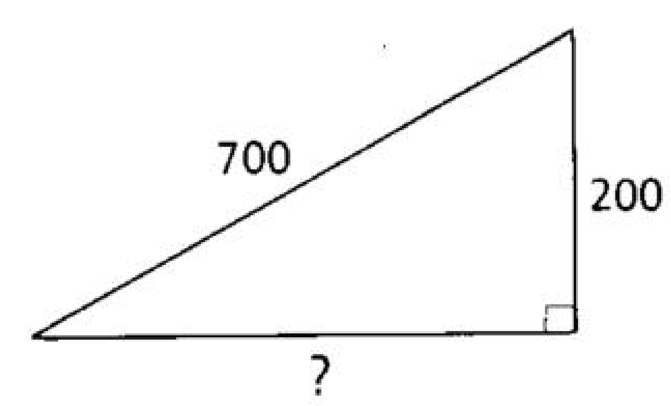


9 Use the reduced-triangle principle to find each missing side.

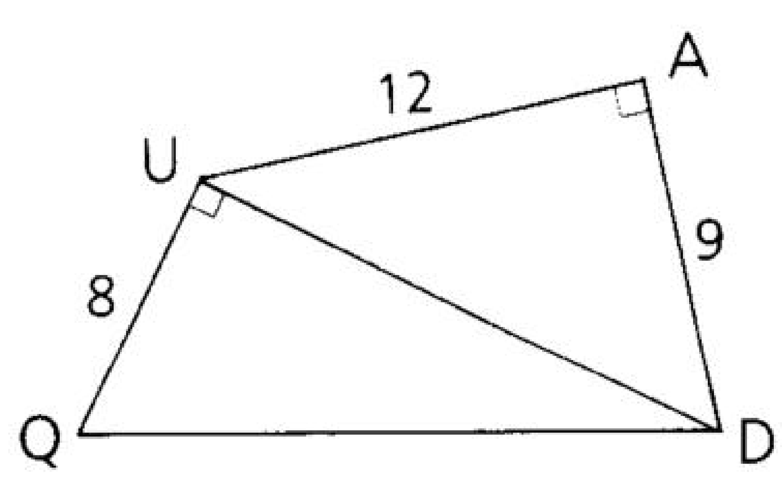
a



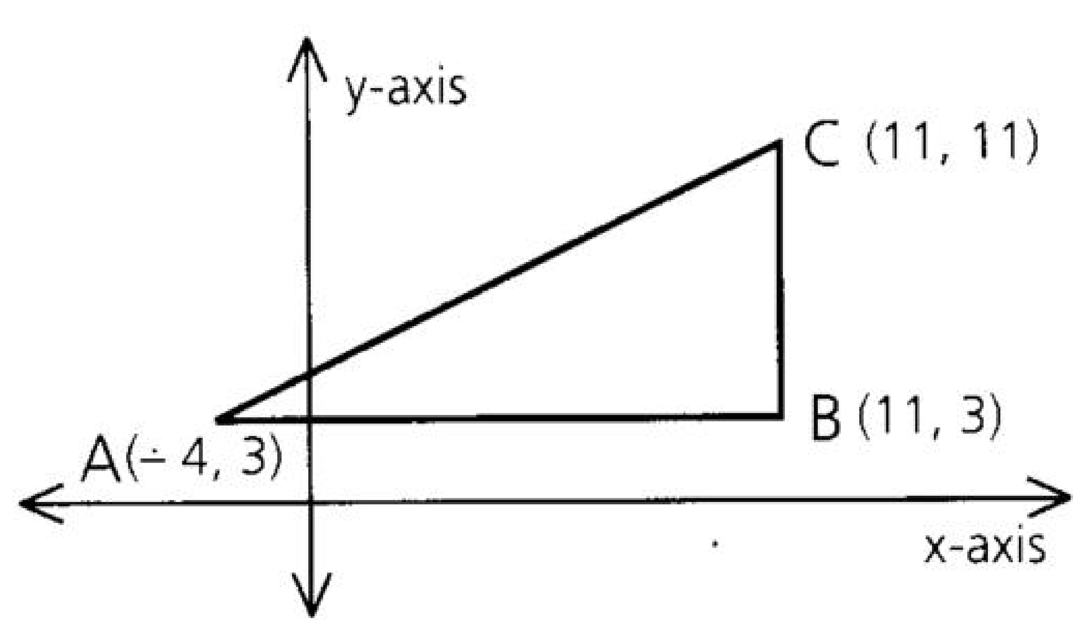
b



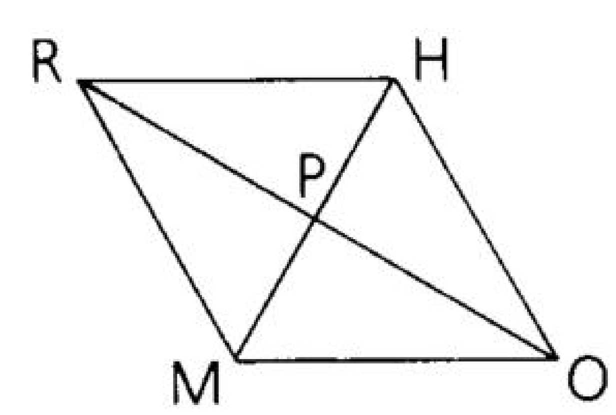
10 Find QD.



11 Find the perimeter and the area of $\triangle ABC$.



12 RHOM is a rhombus with diagonals RO = 48 and HM = 14. Find the perimeter of the rhombus.



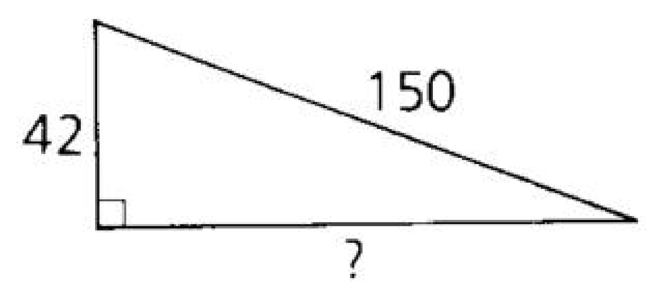
Problem Set B

13 Mary and Larry left the riding stable at 10 A.M. Mary trotted south at 10 kph while Larry galloped east at 16 kph. To the nearest kilometer, how far apart were they at 11:30?

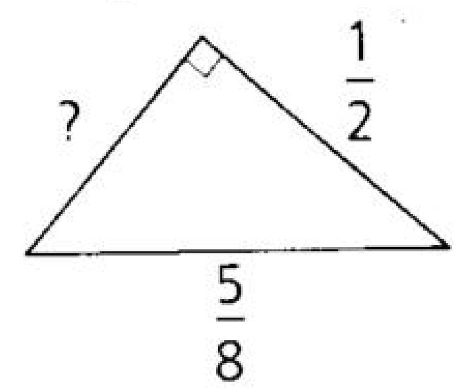
14 Write a coordinate proof to show that the diagonals of a rectangle are congruent.

15 Find the missing side of each triangle.

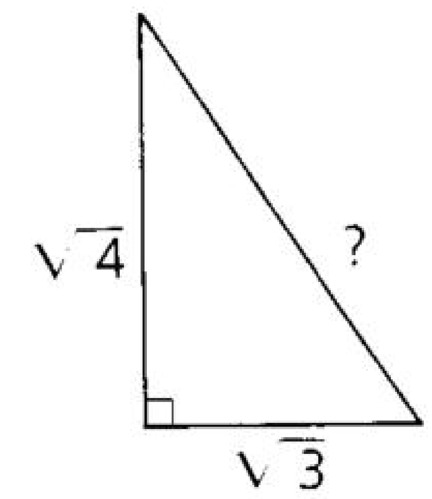
a



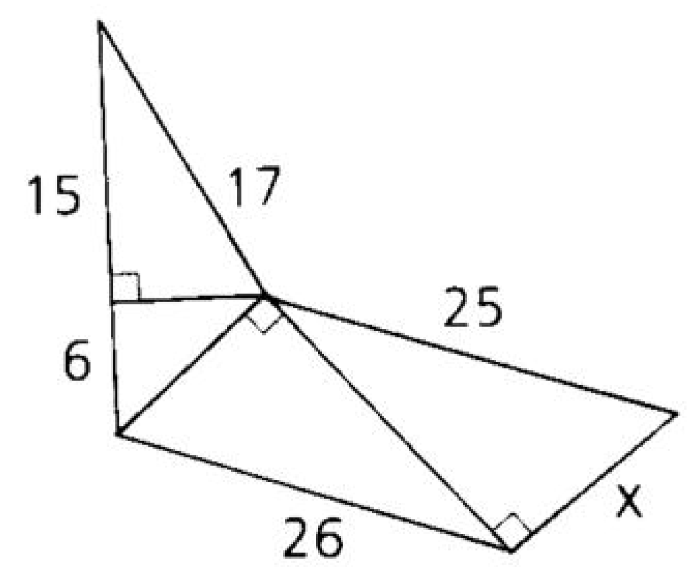
b



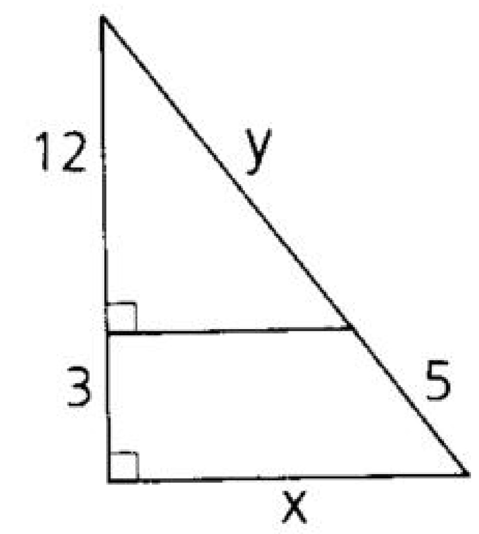
c



16 **a** Find x.



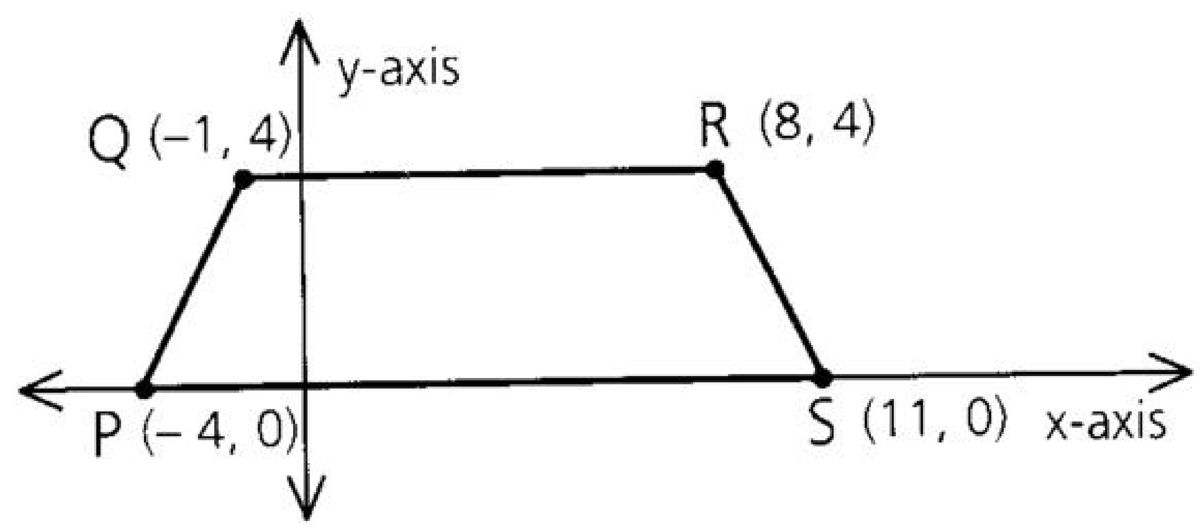
b Find x and y.



17 **a** What is the most descriptive name for quadrilateral PQRS?

b Find the area of PQRS.

c Find PR and QS.

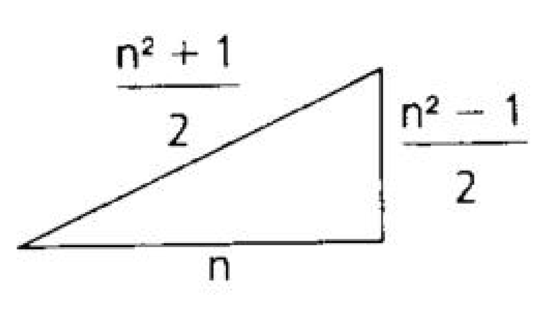


18 A submarine travels an evasive course, trying to outrun a destroyer. It travels 1 km north, then 1 km west, then 1 km north, then 1 km west, and so forth, until it has traveled a total of 41 km. How many kilometers is the sub from the point at which it started?

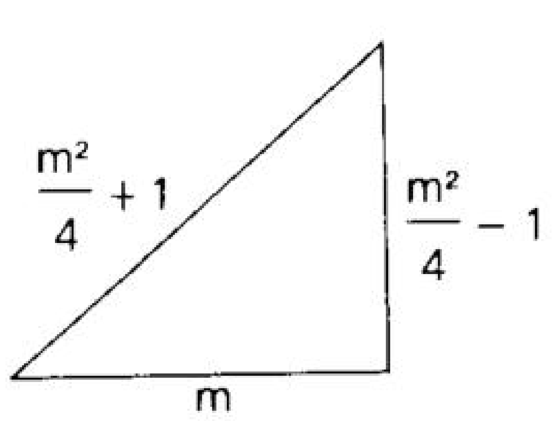
Problem Set C

19 Each of the following is a method for generating sets of whole numbers that represent the sides of a right triangle. Prove that each rule does indeed generate Pythagorean triples.

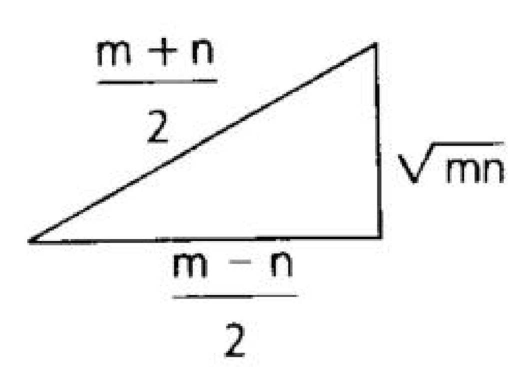
a Rule of Pythagoras
(n is any odd number.)



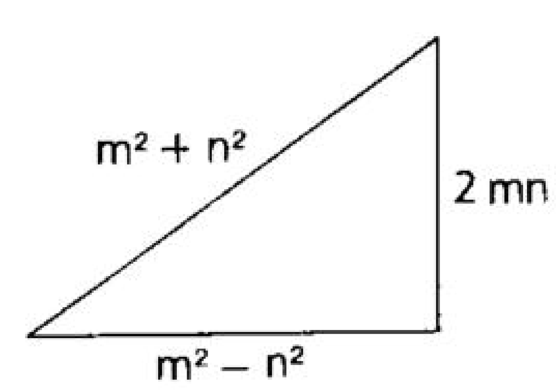
b Rule of Plato
(m is any even number.)



c Rule of Euclid
(m and n are both odd or both even.)



d Rule of Masères
(m and n are any two integers.)



Problem Set C, *continued*

20 Show that the only right triangle in which the lengths of the sides are consecutive integers is the (3, 4, 5) triangle.

21 If a 650-cm ladder is placed against a building at a certain angle, it just reaches a point on the building that is 520 cm above the ground. If the ladder is moved to reach a point 80 cm higher up, how much closer will the foot of the ladder be to the building?

22 The lengths of the legs of a right triangle are x and $3x + y$. The length of the hypotenuse is $4x - y$. Find the ratio of x to y.

23 Six slips of paper, each containing a different one of the numbers 3, 4, 5, 6, 8, and 10, are placed in a hat. Then two of the slips are drawn at random.

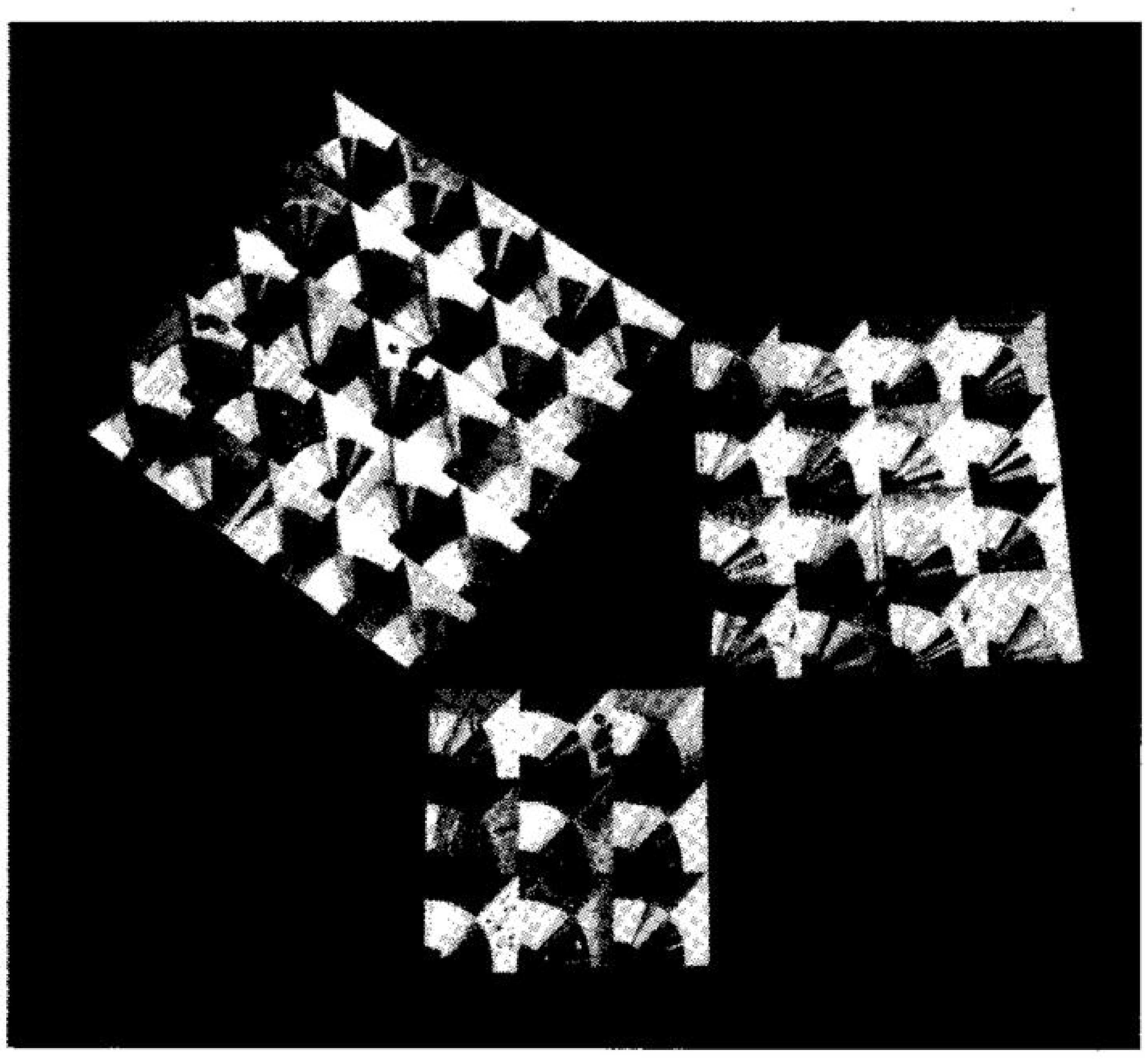
a What is the probability that the numbers drawn are the lengths of two of the sides of a triangle of the (3, 4, 5) family?

b What is the probability that the numbers drawn are lengths of a leg and hypotenuse of a triangle of the (3, 4, 5) family?

Problem Set D

24 Find the length of the hypotenuse of the largest Pythagorean-triple triangle in which 16 is the measure of a leg.

25 Find all right triangles in which one side is 20 and other sides are integral.



9.7 SPECIAL RIGHT TRIANGLES

Objectives

After studying this section, you will be able to

- Identify the ratio of side lengths in a 30°-60°-90° triangle
- Identify the ratio of side lengths in a 45°-45°-90° triangle

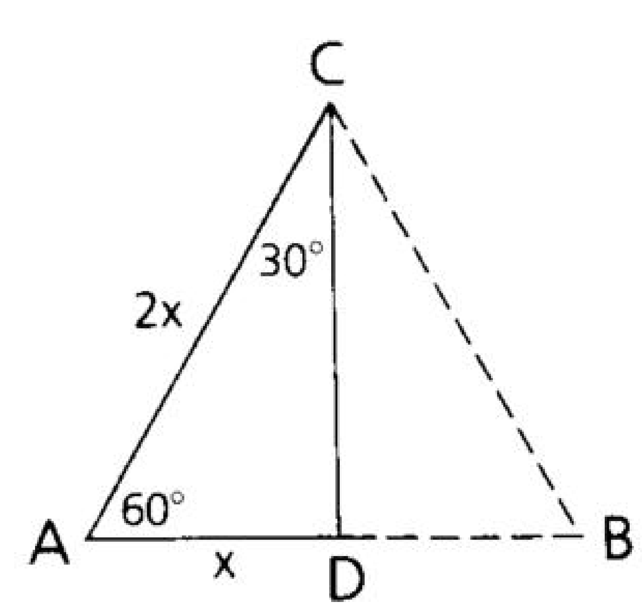
Part One: Introduction

30°-60°-90° Triangles

You will find it useful to know the ratio of the sides of a triangle with angles of 30°, 60°, and 90°.

Theorem 72 ***In a triangle whose angles have the measures* 30, 60, *and* 90, *the lengths of the sides opposite these angles can be represented by* x, $x\sqrt{3}$, *and* $2x$ *respectively.* (30°-60°-90°-*Triangle Theorem*)**

Given: $\triangle ABC$ is equilateral.
$\overrightarrow{CD}$ bisects $\angle ACB$.
Prove: $AD:DC:AC = x:x\sqrt{3}:2x$



Proof: Since $\triangle ABC$ is equilateral, $\angle ACD = 30°$, $\angle A = 60°$, $\angle ADC = 90°$, and $AD = \frac{1}{2}(AC)$.
By the Pythagorean Theorem, in $\triangle ADC$,

$$x^2 + (DC)^2 = (2x)^2$$
$$x^2 + (DC)^2 = 4x^2$$
$$(DC)^2 = 3x^2$$
$$DC = x\sqrt{3}$$

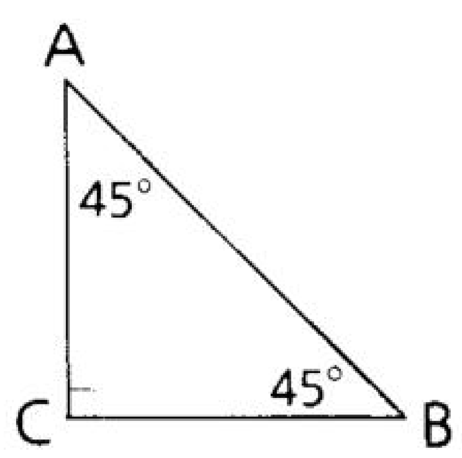
Thus, $AD:DC:AC = x:x\sqrt{3}:2x$

45°-45°-90° Triangles

The sides of a triangle with angles of 45°, 45°, and 90° are also in an easily remembered ratio.

Theorem 73 ***In a triangle whose angles have the measures* 45, 45, *and* 90, *the lengths of the sides opposite these angles can be represented by* x, x, *and* $x\sqrt{2}$, *respectively.* (45°-45°-90°-*Triangle Theorem*)**

Given: $\triangle ACB$, with $\angle A = 45°$ and $\angle B = 45°$.
Prove: $AC:CB:AB = x:x:x\sqrt{2}$
The proof of this theorem is left to you.



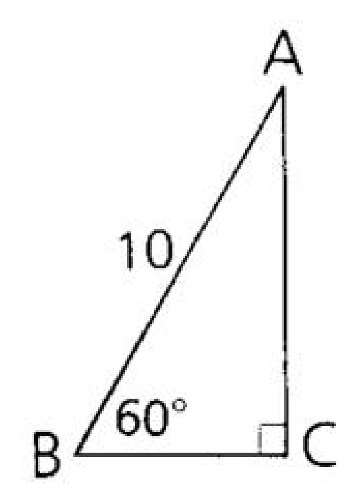
You will see 30°-60°-90° and 45°-45°-90° triangles frequently in this book and in other mathematics courses. Their ratios are worth memorizing now.

Six Common Families of Right Triangles	
30°-60°-90° $\Leftrightarrow$ $(x, x\sqrt{3}, 2x)$	(5, 12, 13)
45°-45°-90° $\Leftrightarrow$ $(x, x, x\sqrt{2})$	(7, 24, 25)
(3, 4, 5)	(8, 15, 17)

Part Two: Sample Problems

Problems 1 and 2 involve 30°-60°-90° triangles. In each, start by placing x on the side opposite (across from) the 30° angle, $x\sqrt{3}$ on the side opposite the 60° angle, and $2x$ on the hypotenuse.

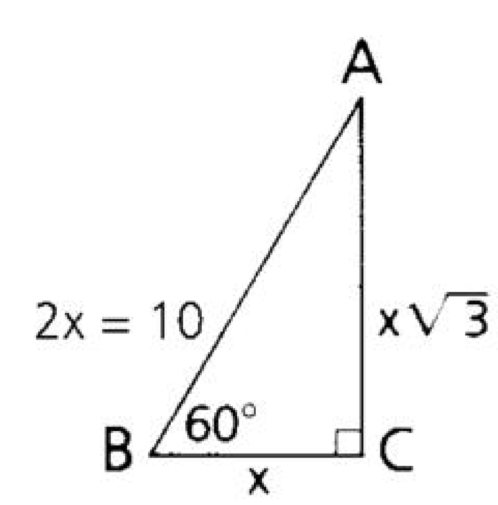
Problem 1 *Type: Hypotenuse* $(2x)$ *known*
Find BC and AC.



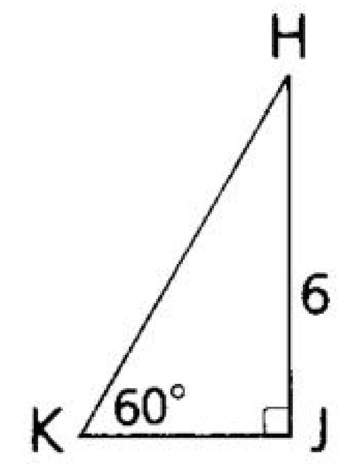
Solution Place x, $x\sqrt{3}$, and $2x$ on a copy of the diagram.

$$2x = 10$$
$$x = 5$$

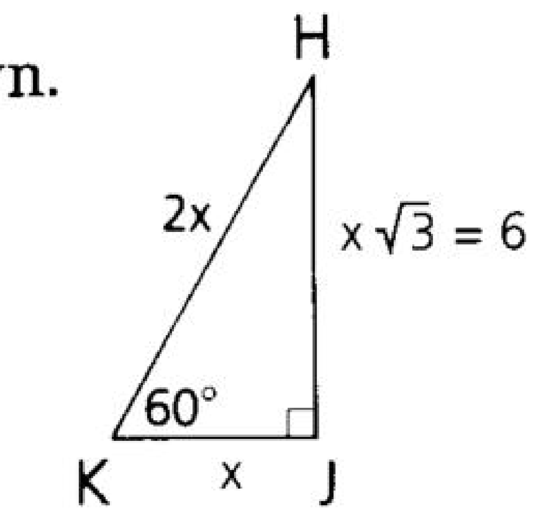
Hence, $BC = 5$, and
$AC = 5\sqrt{3}$



Problem 2 *Type: Longer leg* $(x\sqrt{3})$ *known*
Find JK and HK.



Solution Place x, $x\sqrt{3}$, and 2x on the figure as shown.



$$x\sqrt{3} = 6$$

$$x = \frac{6}{\sqrt{3}}$$

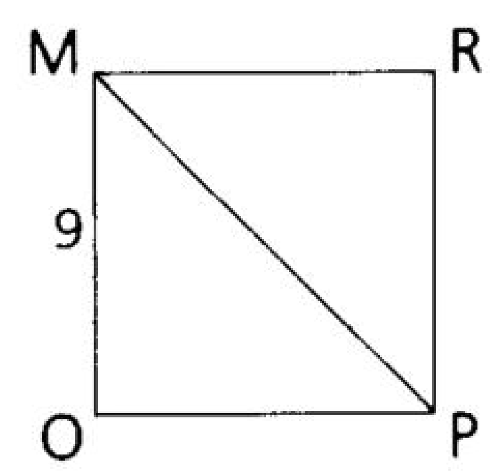
$$= \frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

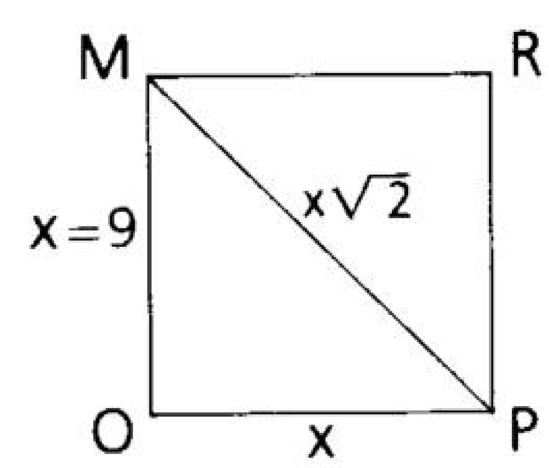
Hence, $JK = 2\sqrt{3}$, and $HK = 2(2\sqrt{3}) = 4\sqrt{3}$.

Problems 3 and 4 involve 45°-45°-90° triangles. In each, start by placing x on each leg and $x\sqrt{2}$ on the hypotenuse.

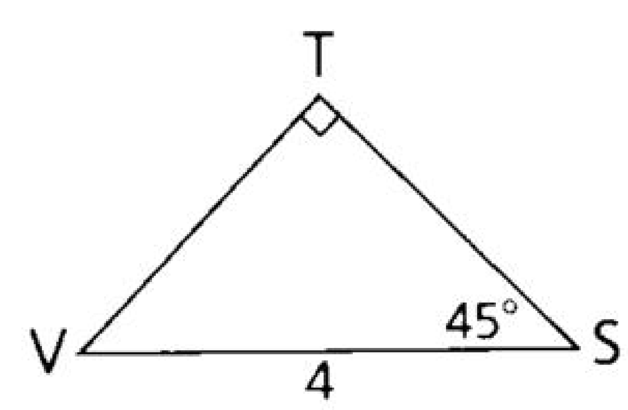
Problem 3 *Type: Leg (x) known*
MOPR *is a square.*
Find MP.



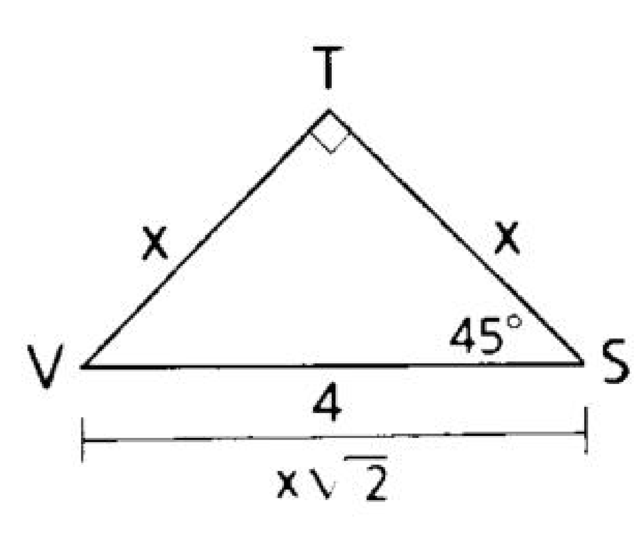
Solution A diagonal divides a square into two 45°-45°-90° triangles.
Place x, x, and $x\sqrt{2}$ as shown.
Since $x = 9$, $MP = 9\sqrt{2}$.



Problem 4 *Type: Hypotenuse* $(x\sqrt{2})$ *known*
Find ST and TV.



Solution Place x, x, and $x\sqrt{2}$ as shown.



$$x\sqrt{2} = 4$$

$$x = \frac{4}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

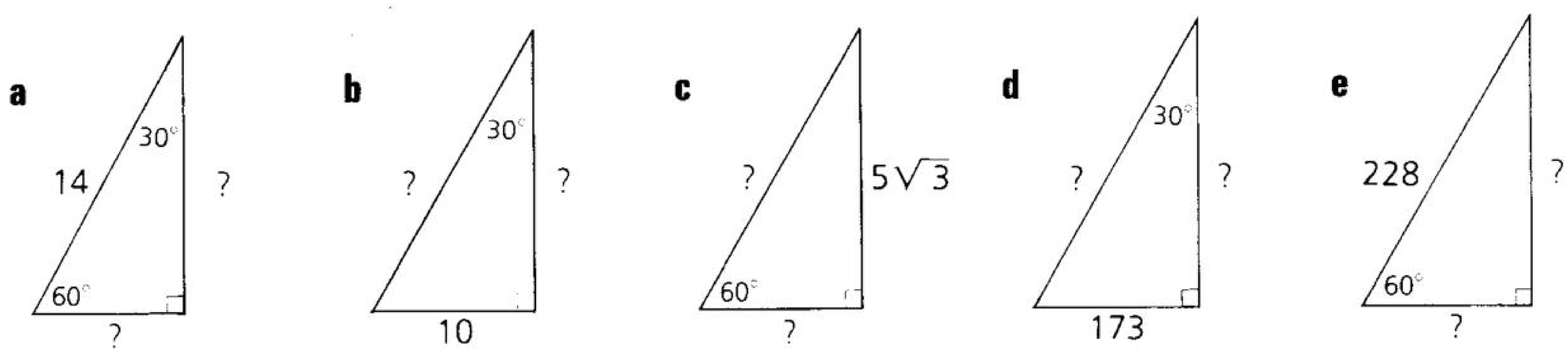
$$= \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

Hence, $ST = TV = 2\sqrt{2}$.

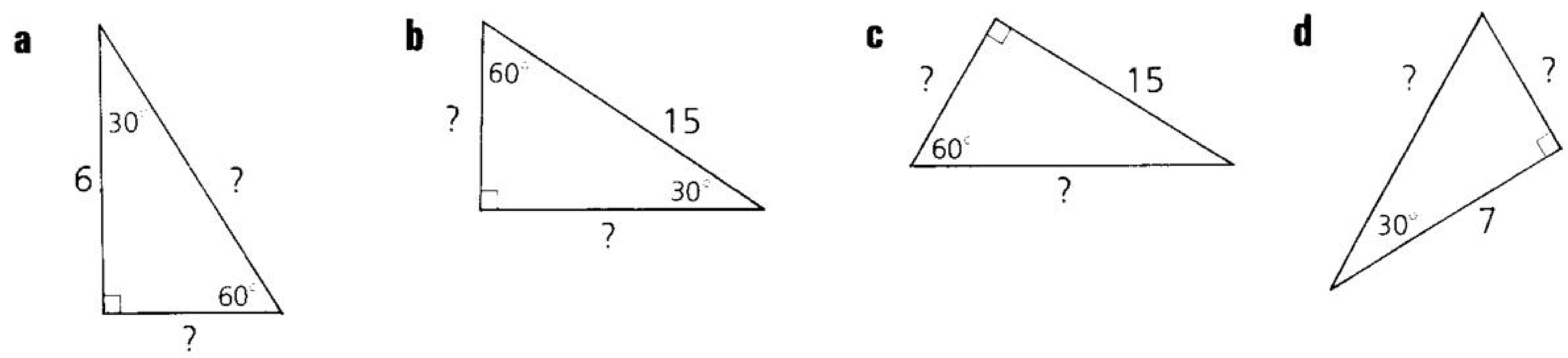
Part Three: Problem Sets

Problem Set A

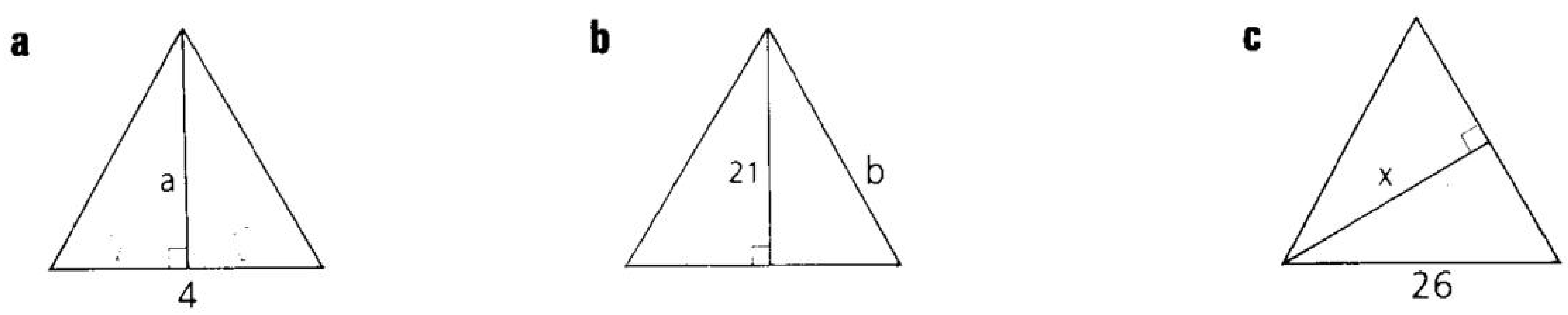
1 Find the two missing sides in each 30°-60°-90° triangle. Try to do the calculations in your head.



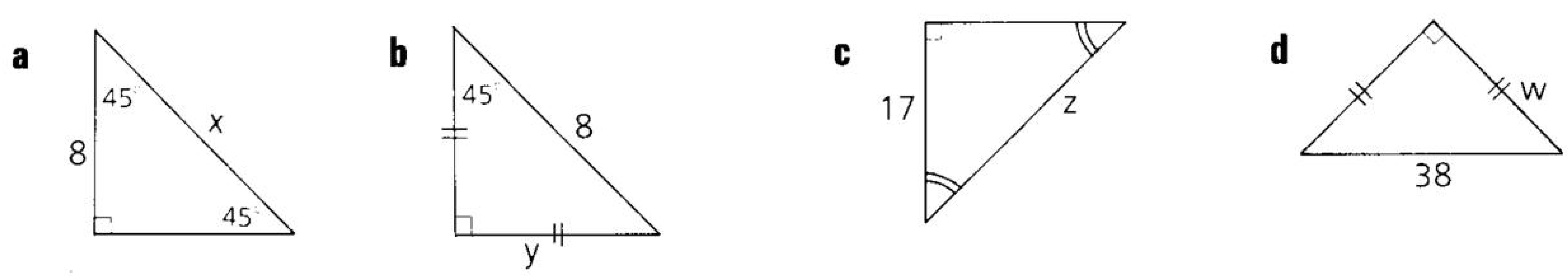
2 Find the two missing sides of each triangle. (Hint: These are a bit harder, and you may want to put x, $x\sqrt{3}$, and 2x on the proper sides as shown in the sample problems.)



3 Solve for the variable in each of these equilateral triangles.

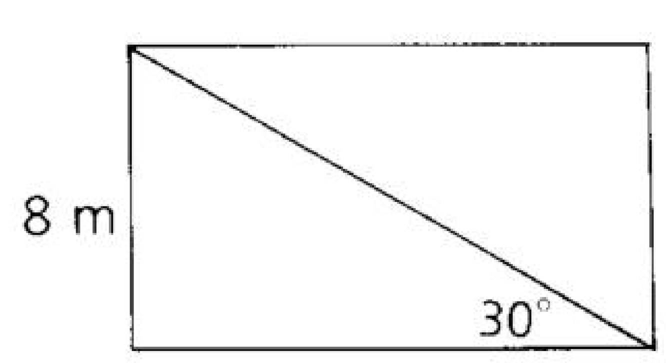


4 Solve for the variable in each of these 45°-45°-90° triangles.



5 The perimeter of a square is 44. Find the length of a diagonal.

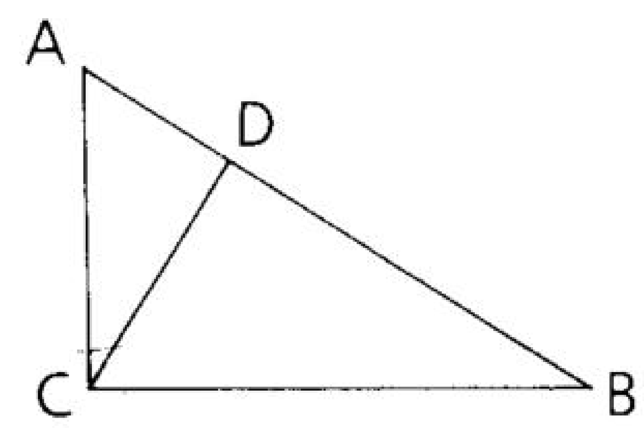
6 Find the length of the diagonal of the rectangle.



7 Find the altitude of an equilateral triangle if a side is 6 mm long.

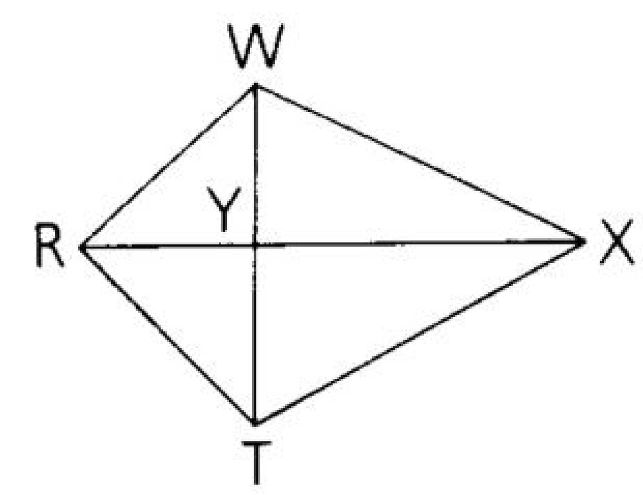
8 Given: $\overline{AC} \perp \overline{BC}$, $\overline{CD} \perp \overline{AB}$,
$\angle B = 30°$, $BC = 8\sqrt{3}$

Find: CD



9 Given: TRWX is a kite ($\overline{TR} \cong \overline{WR}$ and $\overline{TX} \cong \overline{XW}$).
RY = 5, TW = 10, YX = 12

Find: **a** TR
b WX



10 **a** Find the ratio of the longer leg to the hypotenuse in a 30°-60°-90° triangle.

b Find the ratio of one of the legs to the hypotenuse in a 45°-45°-90° triangle.

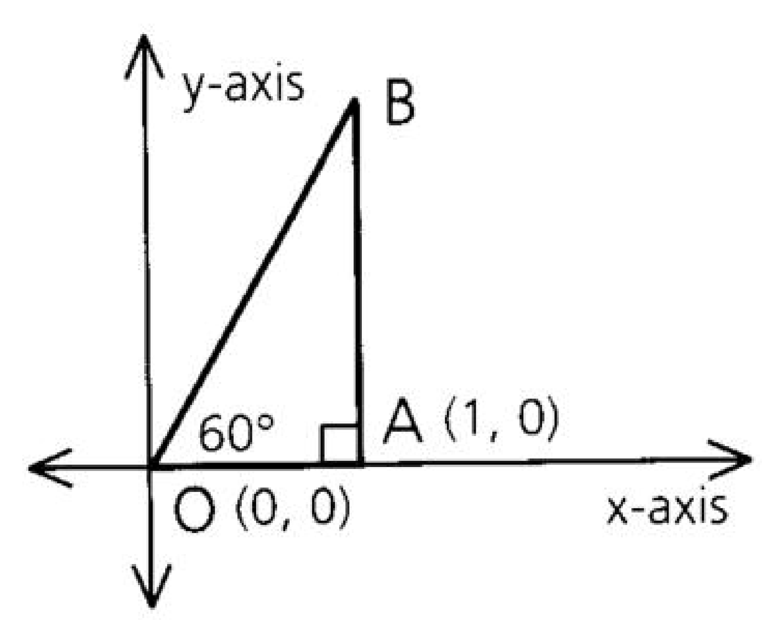
11 Plato is alleged to have said that the 30°-60°-90° triangle was the most beautiful right triangle in the world. Grunts Giraffe, a sophomore student at Animal High, is alleged to have said that the 30°-60°-90° triangle didn't look very pretty to him. Who was Plato, and what do you think he meant by *beautiful?*

Problem Set B

12 **a** Find the coordinates of B.

b Find the slope of $\overleftrightarrow{OB}$.

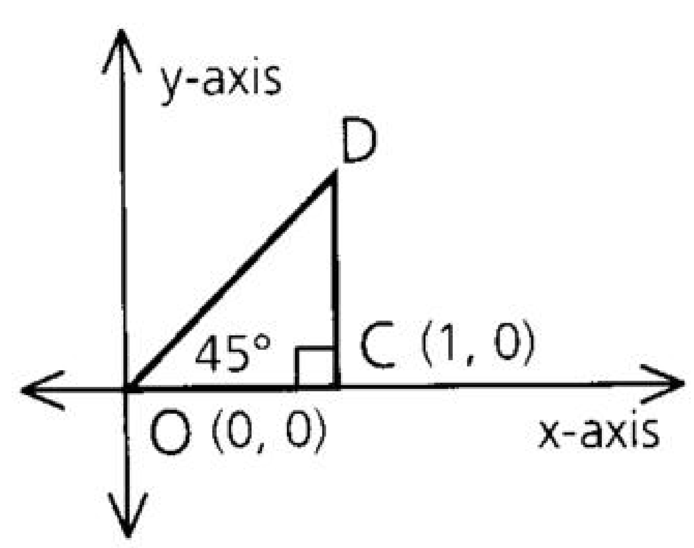
c Find $\frac{AB}{OA}$. (In a trigonometry class, this ratio is called the *tangent* of angle BOA.)



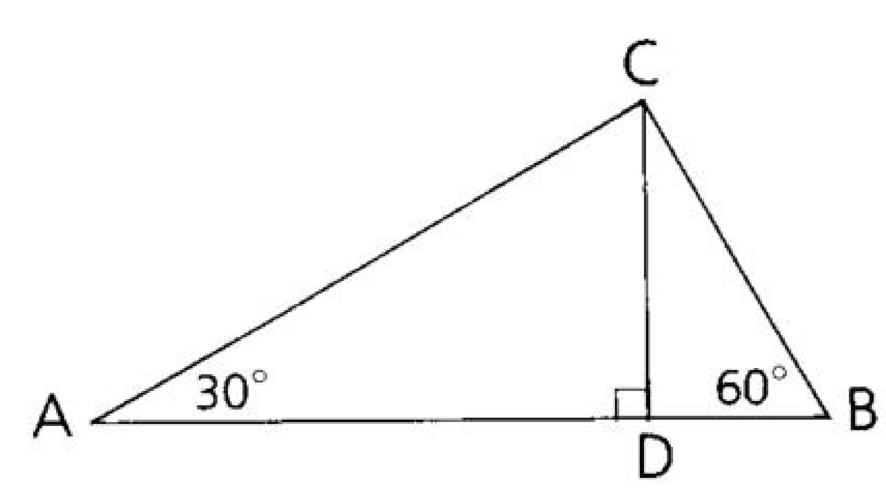
13 **a** Find the coordinates of D.

b Find the slope of $\overleftrightarrow{OD}$.

c Find the tangent of 45°.

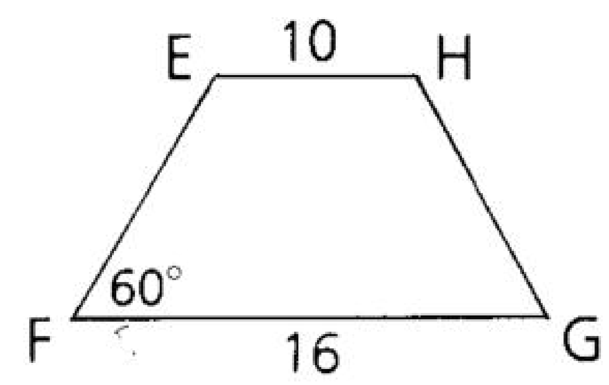


14 Show that in a 30°-60°-90° triangle the altitude to the hypotenuse divides the hypotenuse in the ratio 1:3. (Hint: Let DB = x. Then CD = $x\sqrt{3}$. Now solve for AD.)

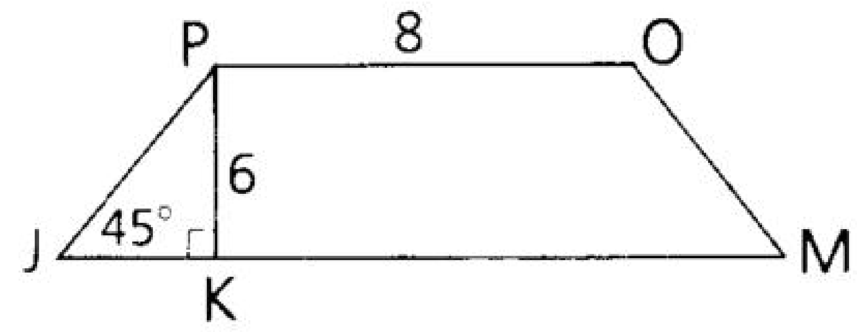


Problem Set B, *continued*

15 Find the perimeter of the isosceles trapezoid EFGH. (Hint: Drop altitudes of the trapezoid from E and H.)



16 Given: $\overline{PK}$ is an altitude of isosceles trapezoid JMOP.
$PK = 6$, $PO = 8$, $\angle J = 45°$
Find: The perimeter of JMOP



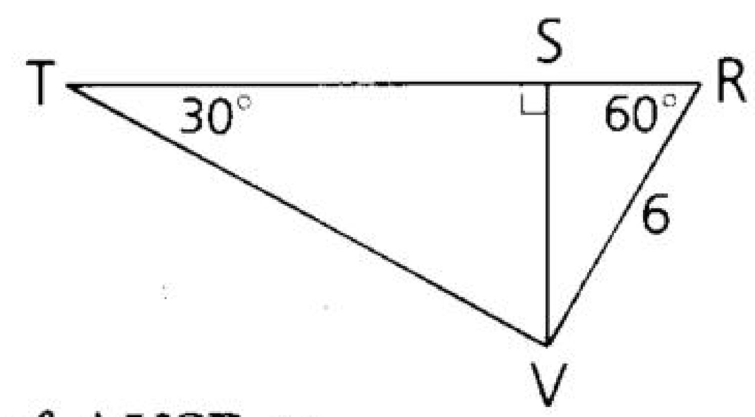
17 Using the figure, find

a VS

b ST

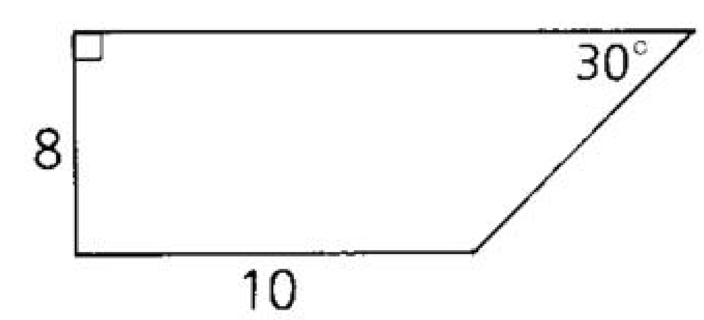
c VT

d The ratio of the perimeter of $\triangle VSR$ to the perimeter of $\triangle VRT$

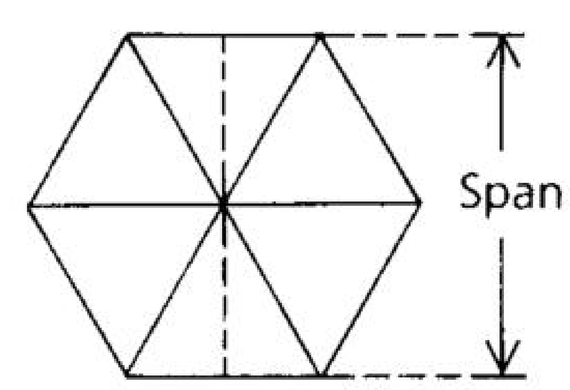


18 One of the angles of a rhombus has a measure of 120. If the perimeter of the rhombus is 24, find the length of each diagonal.

19 Find, to the nearest tenth, the perimeter of the trapezoid.



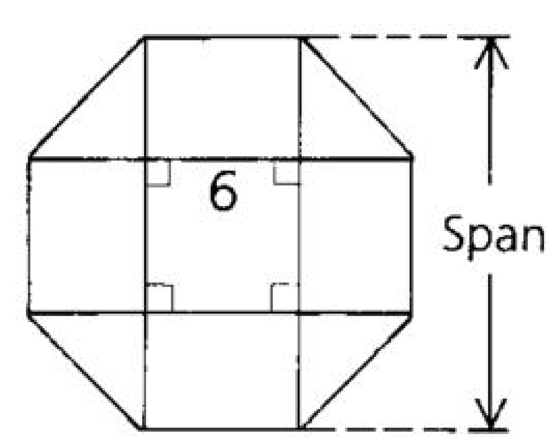
20 Any regular hexagon can be divided into six equilateral triangles by drawing the three diagonals shown. Find the span of a regular hexagon with sides 12 dm long.



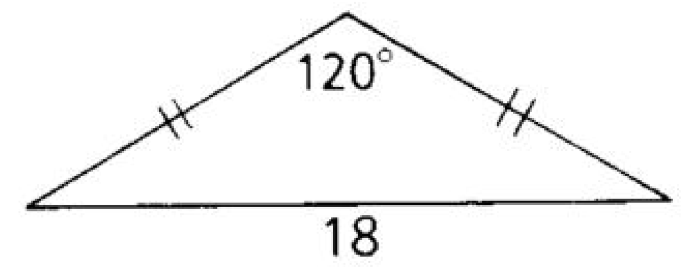
21 Any regular octagon can be divided into rectangles and right triangles. Here, a side of the central square is 6 units long.

a Find the perimeter of the octagon.

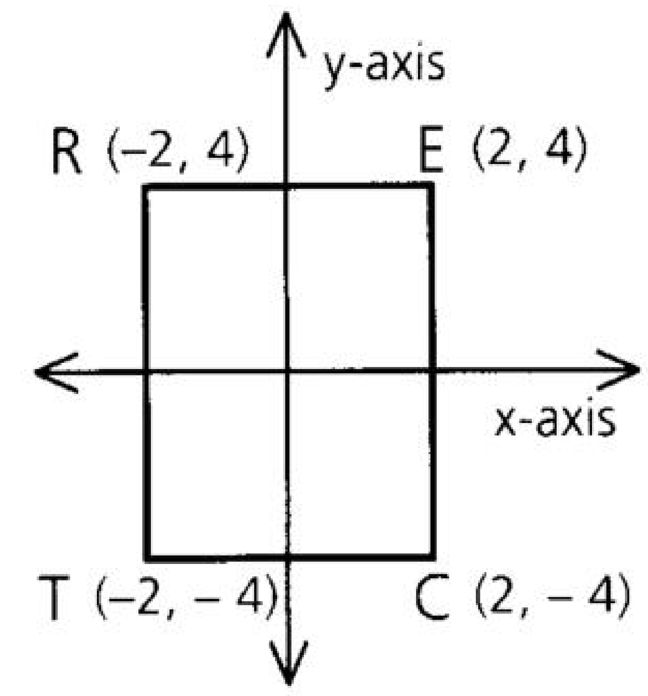
b Find the span of the octagon.



22 Find the altitude to the base of the isosceles triangle shown.

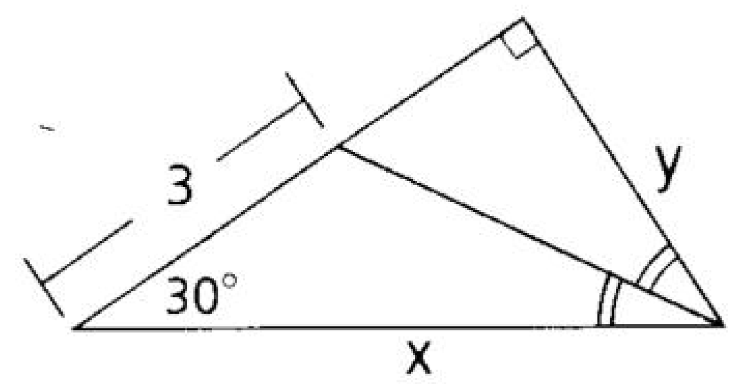


23 If rectangle RECT is rotated about the origin until E lies on the positive y-axis, what will the new coordinates of E be?



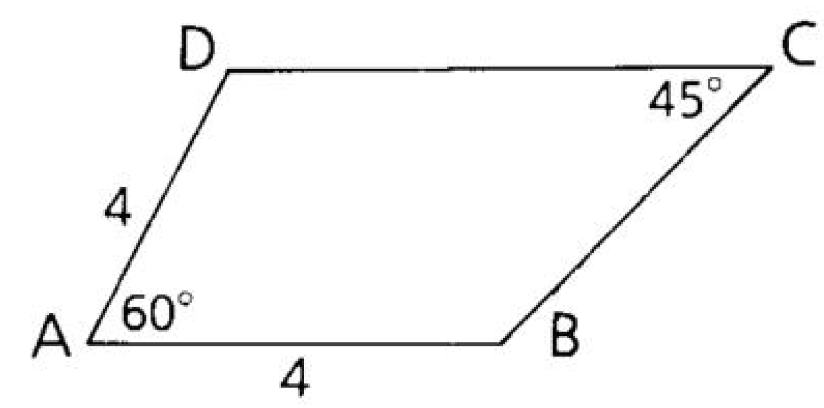
Problem Set C

24 Find x and y.

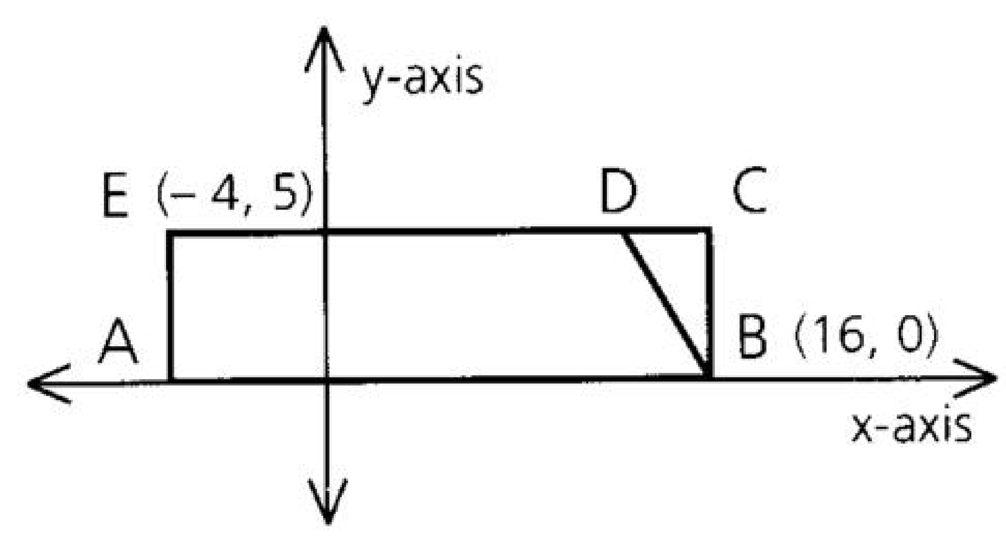


25 Given: ABCD is a trapezoid ($\overline{DC} \parallel \overline{AB}$).
AB = AD = 4,
∠A = 60°, ∠C = 45°

Find: **a** DC
b BC



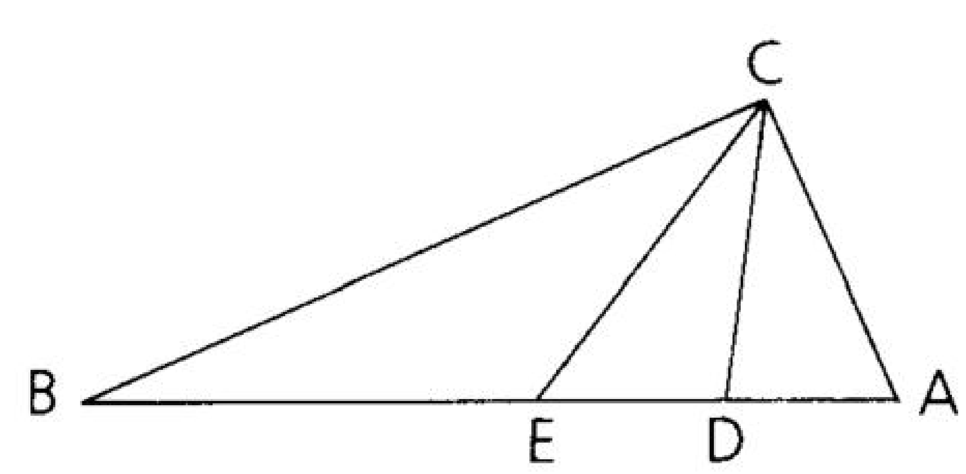
26 If the area of rectangle ABCE is eight times that of △BCD, how far is D from the origin?



Problem Set D

27 Given: ∠ACB is a right angle.
$\overrightarrow{CD}$ and $\overrightarrow{CE}$ trisect ∠ACB.
AC = 5, BC = 12

Find: CE (Hint: Draw a perpendicular from E to $\overline{CB}$.)



In solving probability problems, a tree diagram is sometimes helpful. Consider the following problem:

A bag contains seven red marbles, two blue marbles, and a white marble. A woman reaches into the bag and draws two marbles.

a What is the probability that she has drawn two red marbles?

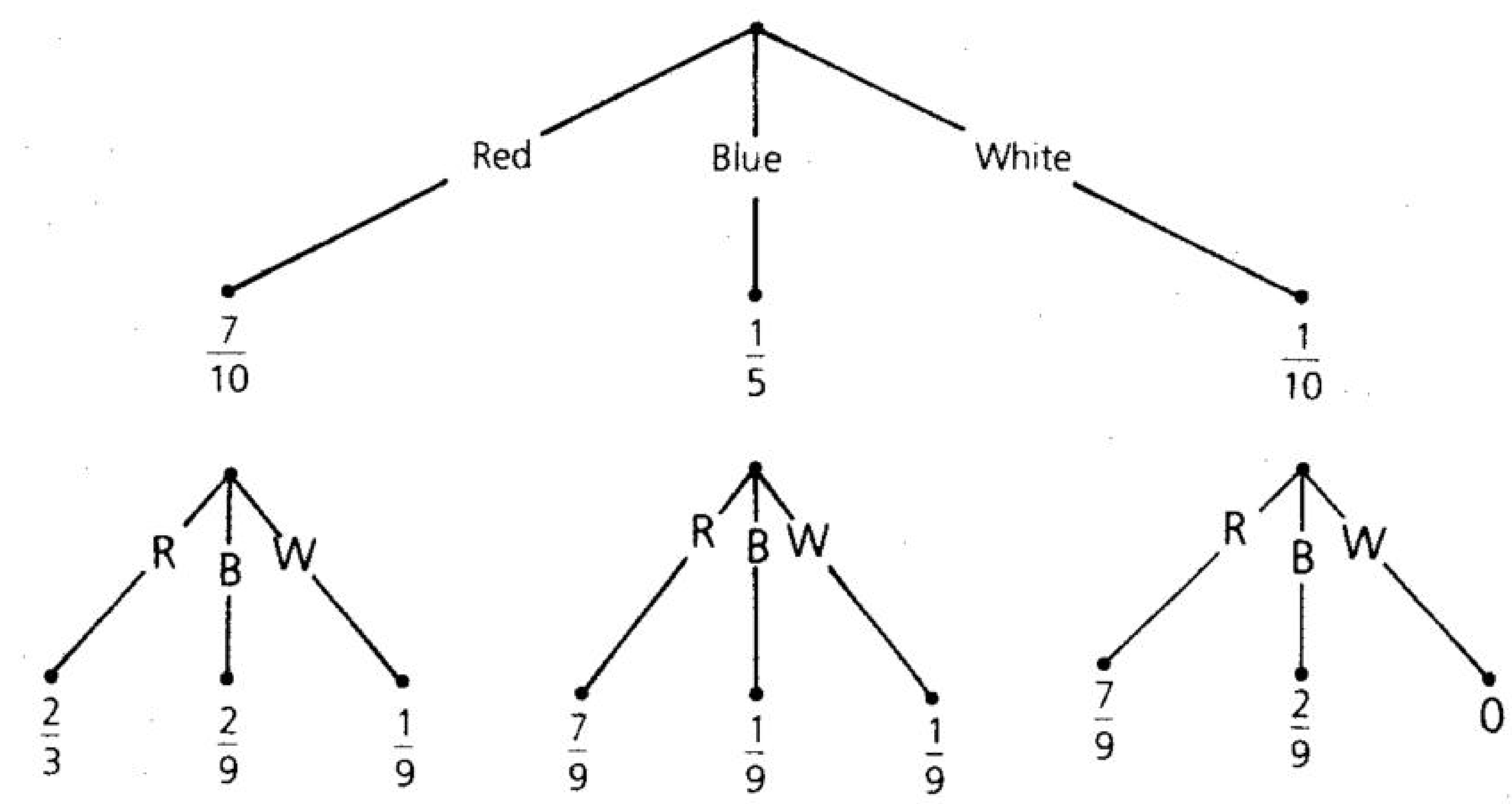
b What is the probability that she has drawn one or more red marbles?

Solution:

a The tree diagram below shows that the probability of drawing a red marble and then another red marble is $\frac{7}{10} \cdot \frac{2}{3} = \frac{7}{15}$. So RR $= \frac{7}{15}$.
What are the probabilities of the other eight possible outcomes?

First marble (10 marbles to choose from)

Second marble (9 marbles to choose from)



b The probability of drawing one or more red marbles is the sum of the probabilities of RR, RB, RW, BR, and WR, or $\frac{14}{15}$.

28 Use a tree diagram to solve the following problem:

A bag contains eight right triangles. Five are members of the (3, 4, 5) family, and two are 30°-60°-90° triangles. A puppy falls over the bag, and two triangles fall out on the floor.

a What is the probability that both are members of the (3, 4, 5) family?

b What is the probability that at least one of the triangles is a member of the (3, 4, 5) family?

c What is the probability that one is a member of the (3, 4, 5) family and the other is a 30°-60°-90° triangle?

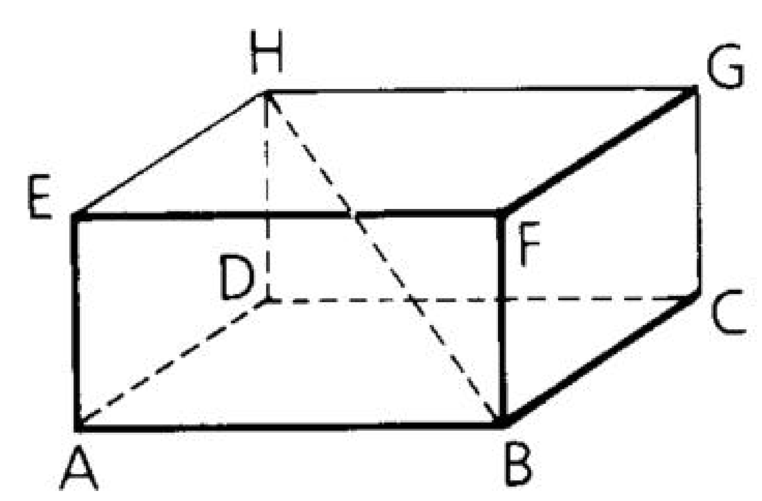
9.8 THE PYTHAGOREAN THEOREM AND SPACE FIGURES

Objective

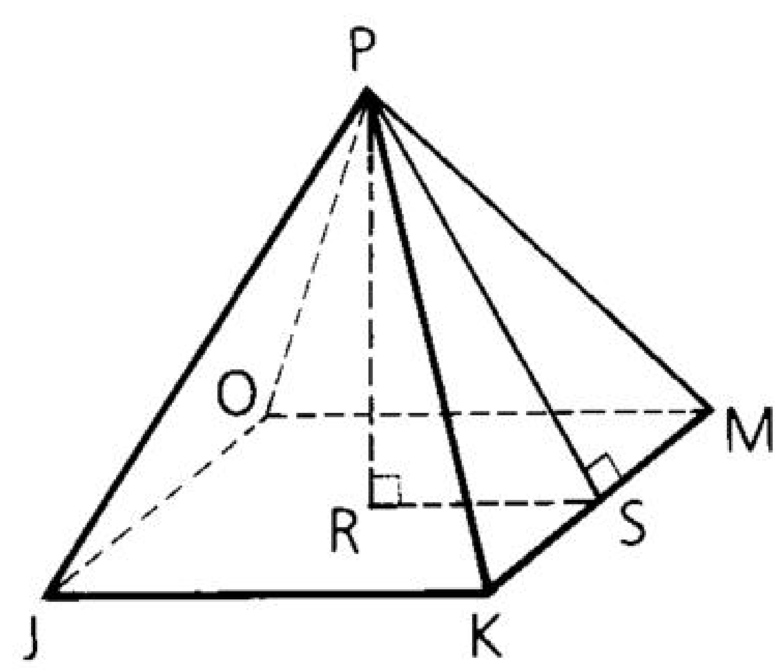
After studying this section, you will be able to

- Apply the Pythagorean Theorem to solid figures

Part One: Introduction



Rectangular Solid



Regular Square Pyramid

Many of the problems in this section will involve the two figures shown above.

In the rectangular solid:

ABFE is one of the 6 rectangular ***faces***

$\overline{AB}$ is one of the 12 ***edges***

$\overline{HB}$ is one of the 4 ***diagonals*** of the solid. (The others are $\overline{AG}$, $\overline{CE}$, and $\overline{DF}$.)

In the regular square pyramid:

JKMO is a square, and it is called the ***base***

P is the ***vertex***

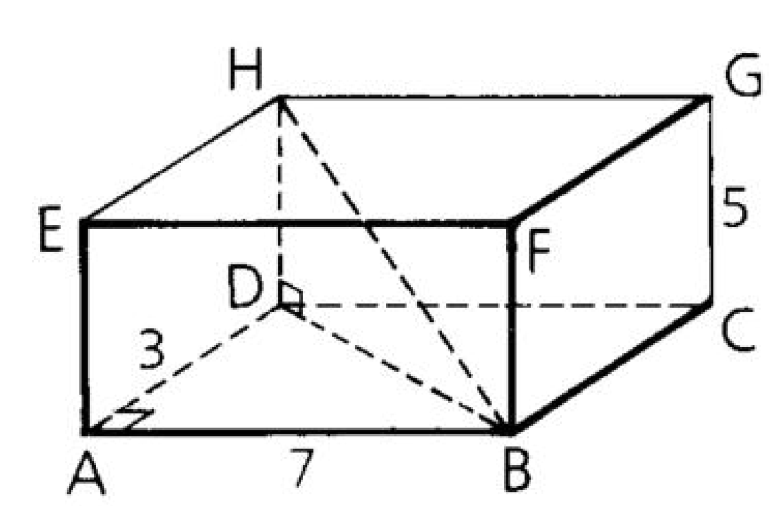
$\overline{PR}$ is the ***altitude*** of the pyramid and is perpendicular to the base at its center.

$\overline{PS}$ is called a ***slant height*** and is perpendicular to a side of the base.

Note A ***cube*** is a rectangular solid in which all edges are congruent.

Part Two: Sample Problems

Problem 1 *The dimensions of a rectangular solid are 3, 5, and 7. Find the diagonal.*



Solution It does not matter which edges are given the lengths 3, 5, and 7. Let AD = 3, AB = 7, and HD = 5, and use the Pythagorean Theorem twice.

In $\triangle ABD$,

$$3^2 + 7^2 = (DB)^2$$
$$9 + 49 = (DB)^2$$
$$\sqrt{58} = DB$$

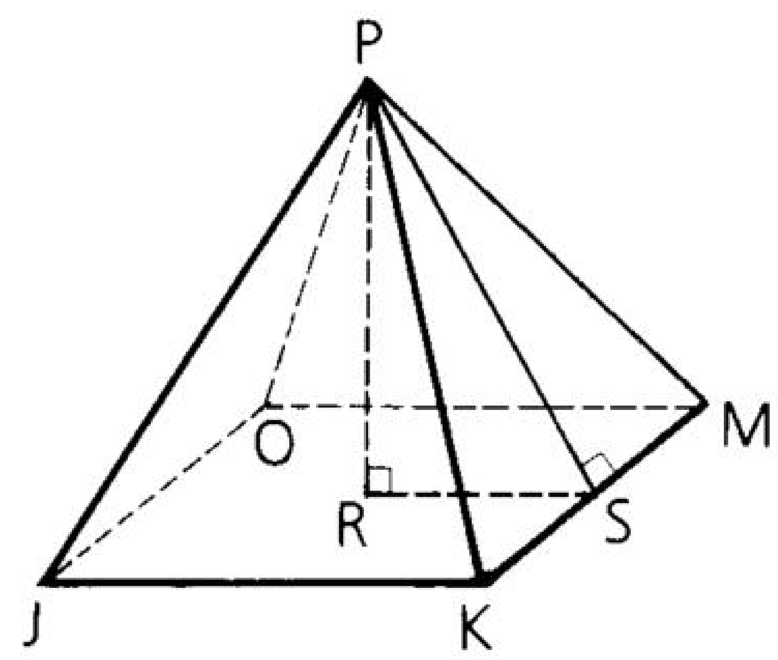
In $\triangle HDB$,

$$5^2 + (\sqrt{58})^2 = (HB)^2$$
$$25 + 58 = (HB)^2$$
$$\sqrt{83} = HB$$

The measure of the diagonal is $\sqrt{83}$.

Problem 2 *Given: The regular square pyramid shown, with altitude $\overline{PR}$ and slant height $\overline{PS}$, perimeter of JKMO = 40, PK = 13*

Find: **a** JK **b** PS **c** PR



Solution

a $JK = \frac{1}{4}(40) = 10$

b The slant height of the pyramid is the $\perp$ bis. of $\overline{MK}$, so PSK is a right $\triangle$.

$$(SK)^2 + (PS)^2 = (PK)^2$$
$$5^2 + (PS)^2 = 13^2$$
$$PS = 12$$

c The altitude of a regular pyramid is perpendicular to the base at its center. Thus, $RS = \frac{1}{2}(JK) = 5$, and PRS is a right $\triangle$.

$$(RS)^2 + (PR)^2 = (PS)^2$$
$$5^2 + (PR)^2 = 12^2$$
$$25 + (PR)^2 = 144$$
$$PR = \sqrt{119}$$

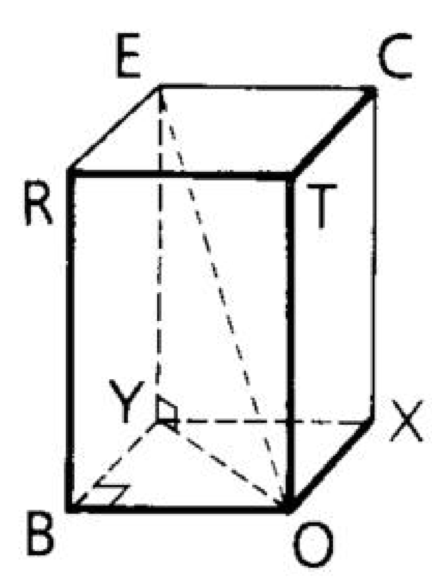
Part Three: Problem Sets

Problem Set A

1 Given: The rectangular solid shown, BY = 3, OB = 4, EY = 12

Find: **a** YO, a diagonal of face BOXY

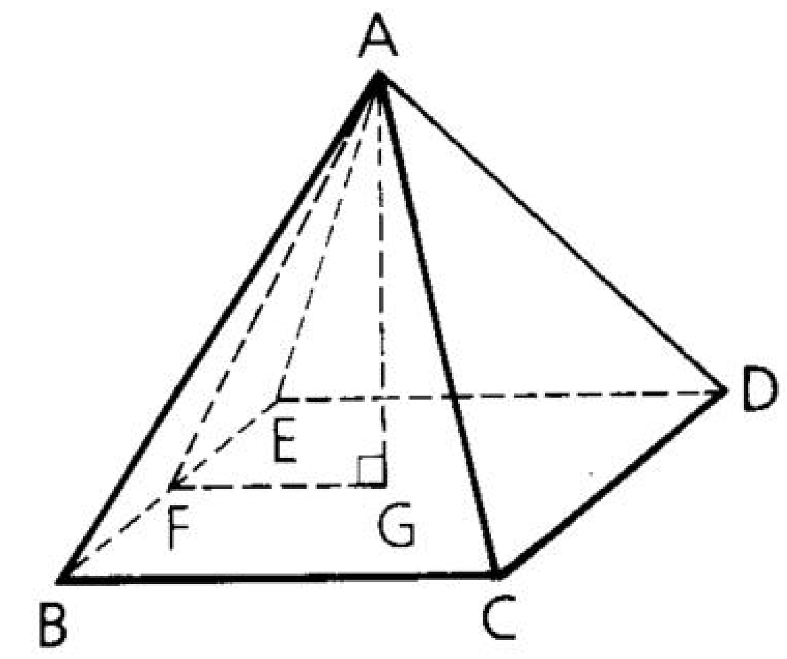
b EO, a diagonal of the solid



2 Find the diagonal of a rectangular solid whose dimensions are 3, 4, and 5.

3 Given: Regular square pyramid ABCDE, with slant height $\overline{AF}$, altitude $\overline{AG}$, and base BCDE; perimeter of BCDE = 40, $\angle AFG = 60°$

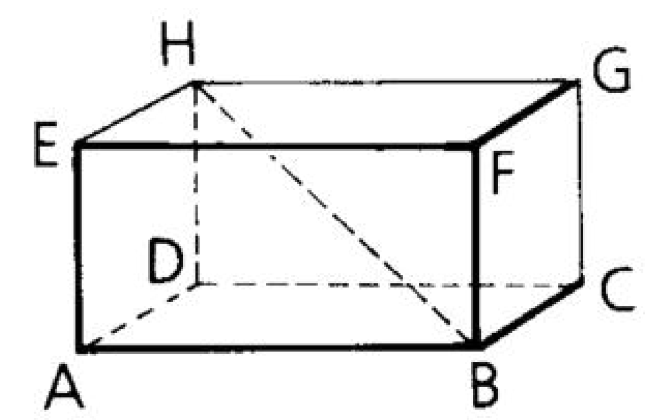
Find: The altitude and the slant height



4 Given: The rectangular solid shown, GC = 8, HG = 12, BC = 9

Find: **a** HB, a diagonal of the solid

b AG, another diagonal of the solid



5 Given: The regular square pyramid shown, with altitude $\overline{PY}$ and slant height $\overline{PR}$, ID = 14, PY = 24

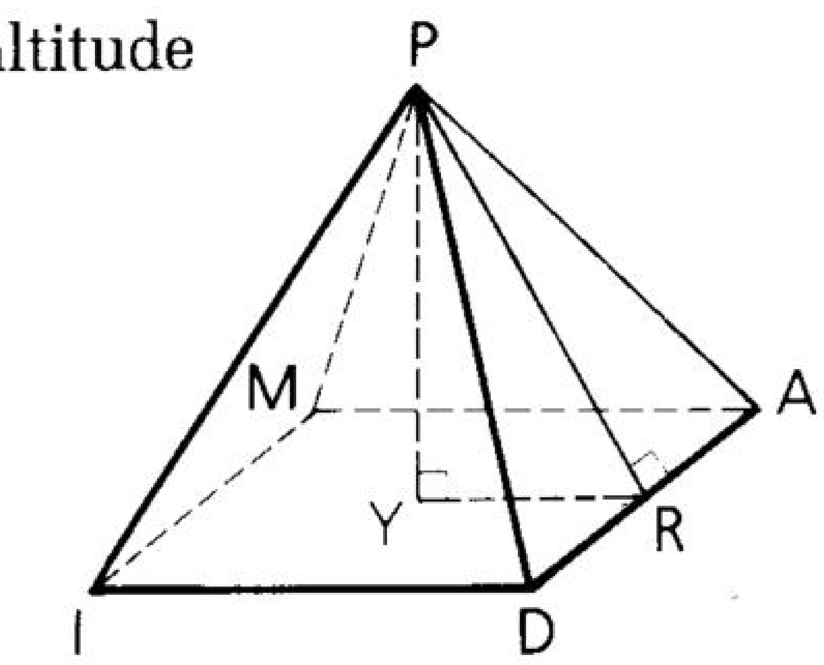
Find: **a** AD

b YR

c PR

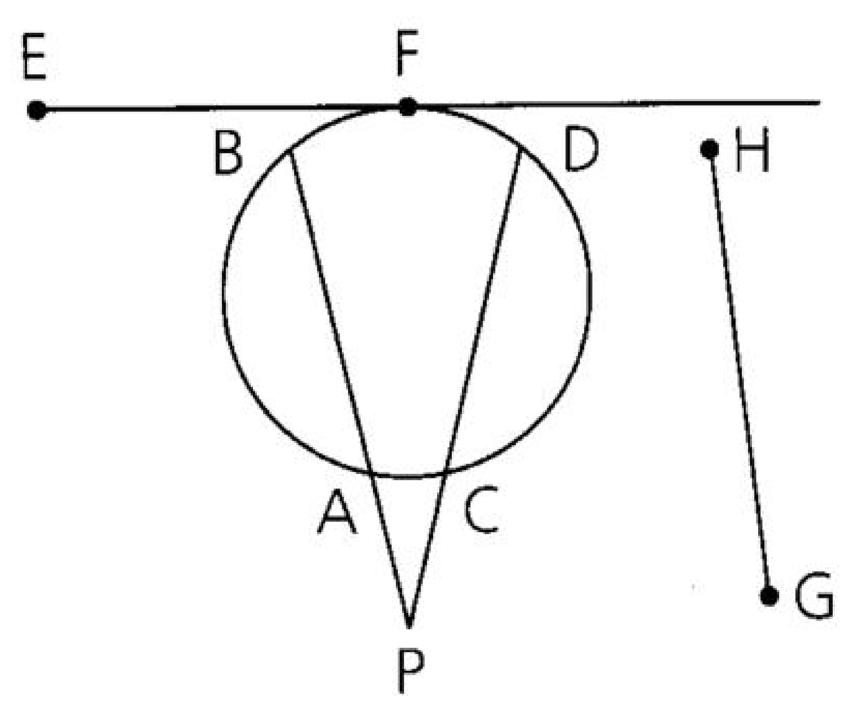
d The perimeter of base AMID

e A diagonal of the base (not shown in the diagram)



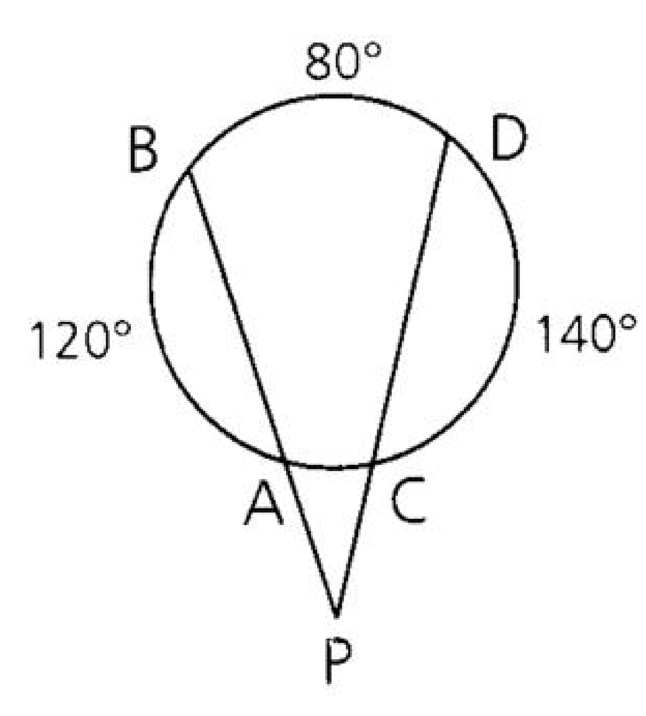
6 Find the slant height of a regular square pyramid if the altitude is 12 and one of the sides of the square base is 10.

7 A line that intersects a circle at two points is called a *secant*. Which of the four lines in the diagram ($\overleftrightarrow{EF}$, $\overleftrightarrow{PB}$, $\overleftrightarrow{PD}$, and $\overleftrightarrow{GH}$) are secants?



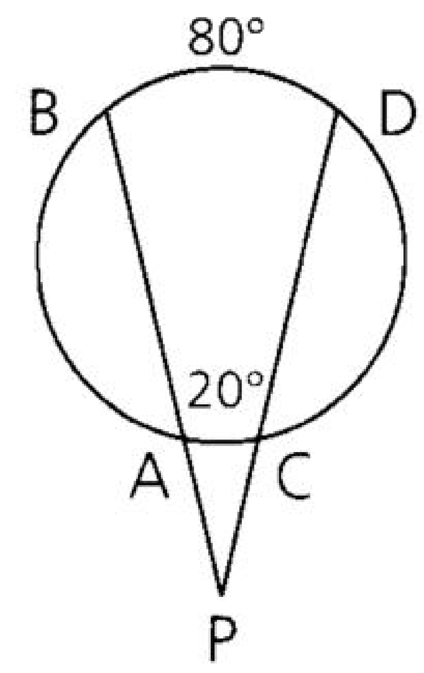
8 Given: Diagram as marked

Find: $m\overset{\frown}{AC}$

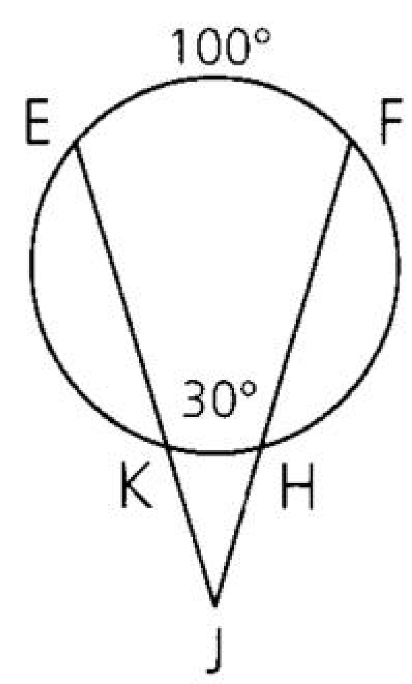


Problem Set A, *continued*

9 Daffy Difference looked ahead to Chapter 10 and found that the measure of a secant-secant angle (such as ∠BPD) is one-half the difference of its two intercepted arcs. Use this information to find m∠BPD.

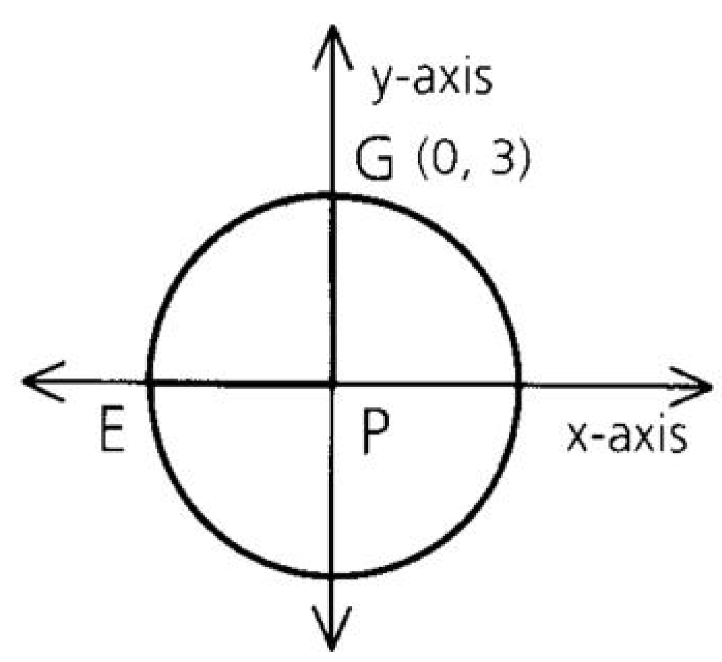


10 Given: Diagram as marked
Find: m∠EJF

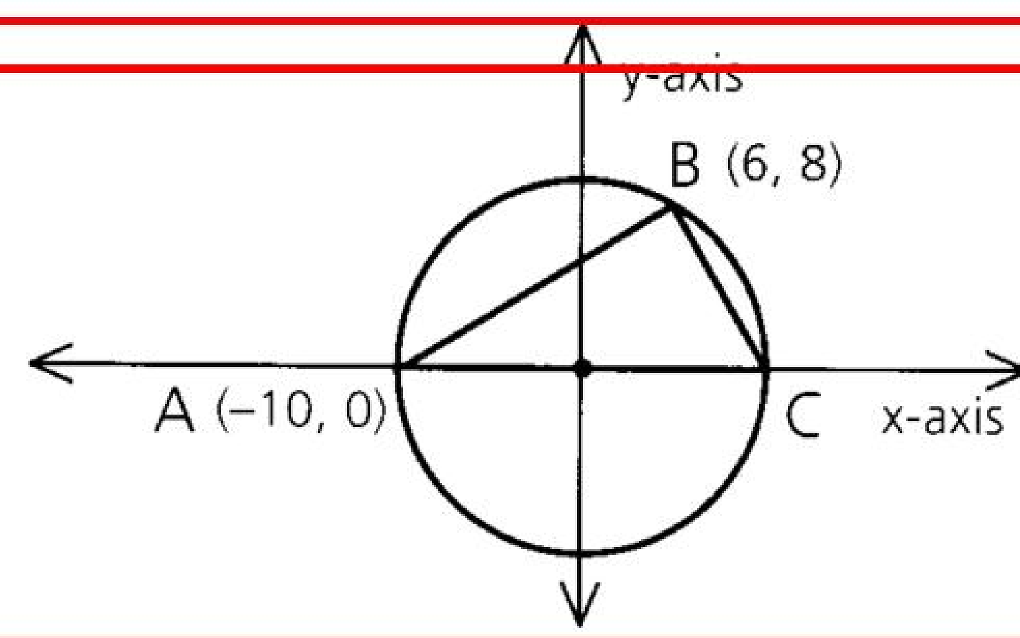


Problem Set B

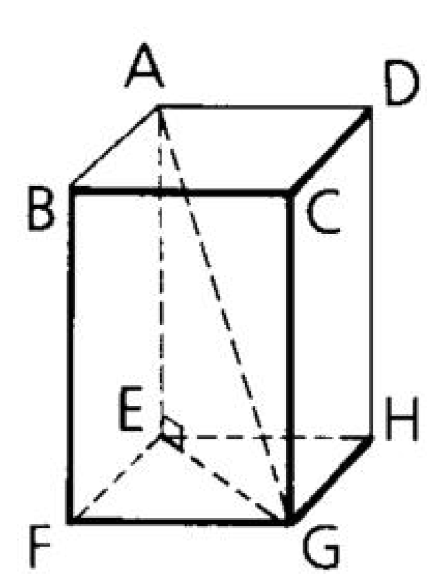
11 Given: ⊙P as shown
Find: **a** The coordinates of point E
b The area of sector EPG to the nearest tenth
c The length of $\overset{\frown}{GE}$ to the nearest tenth



12 Given: Diagram as marked
Find: AB (the length of $\overline{AB}$)



13 ABCDEFGH is a rectangular solid.
a If face diagonal $\overline{CH}$ measures 17, edge $\overline{GH}$ measures 8, and edge $\overline{FG}$ measures 6, how long is diagonal $\overline{AG}$?
b If diagonal $\overline{AG}$ measures 50, edge $\overline{AE}$ measures 40, and edge $\overline{EF}$ measures 3, how long is edge $\overline{FG}$?



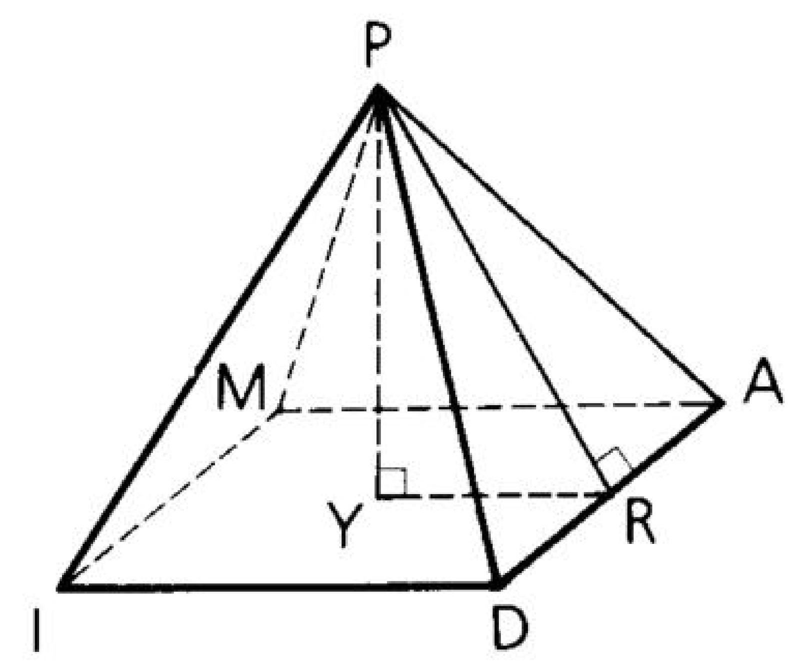
14 PADIM is a regular square pyramid. Slant height $\overline{PR}$ measures 10, and the base diagonals measure $12\sqrt{2}$.

a Find ID.

b Find the altitude of the pyramid.

c Find RD.

d Find PD (length of a lateral edge).



15 Find the diagonal of a cube if each edge is 2.

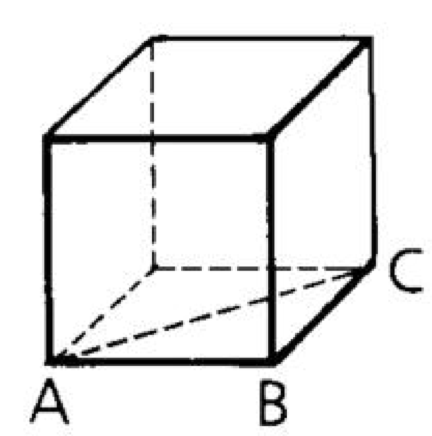
16 Find the diagonal of a cube if the perimeter of a face is 20.

17 The perimeter of the base of a regular square pyramid is 24. If the slant height is 5, find the altitude.

Problem Set C

18 In the cube, find the measure of the diagonal in terms of x if

a AB = x **b** AC = x



19 Find a formula for the length of a diagonal of a rectangular solid. (Use a, b, and c for the three dimensions.)

20 The dimensions of a rectangular solid are in the ratio 3:4:5. If the diagonal is $200\sqrt{2}$, find the three dimensions.

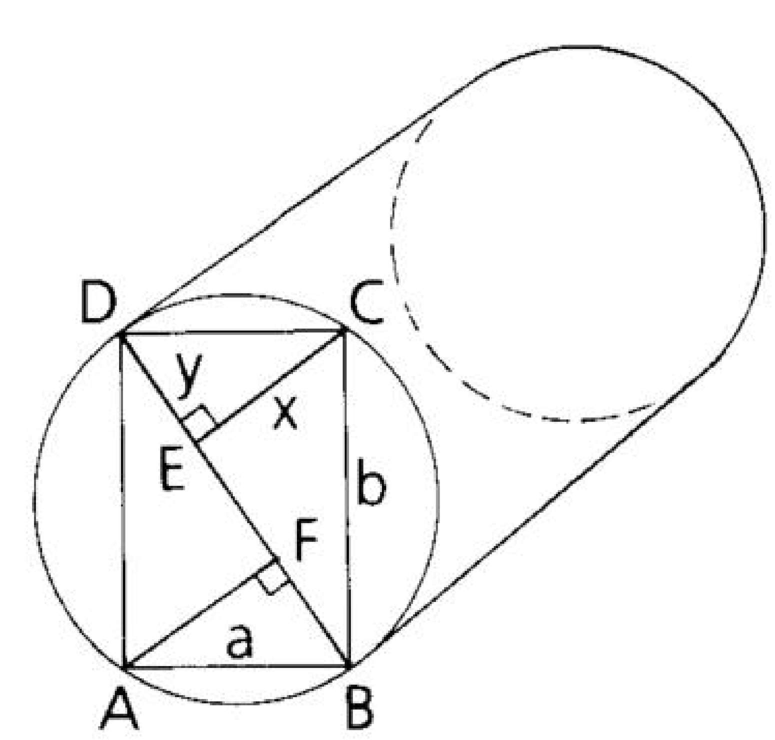
21 The face diagonals of a rectangular box are 2, 3, and 6. Find the diagonal of the box.

22 A pyramid is formed by assembling four equilateral triangles and a square having sides 6 cm long. Find the altitude and the slant height.

Problem Set D

23 The strongest rectangular beam that can be cut from a circular log is one having a cross section in which the diagonal joining two vertices is trisected by perpendicular segments dropped from the other vertices.

If AB = a, BC = b, CE = x, and DE = y, show that $\frac{b}{a} = \frac{\sqrt{2}}{1}$.



9.9 INTRODUCTION TO TRIGONOMETRY

Objective

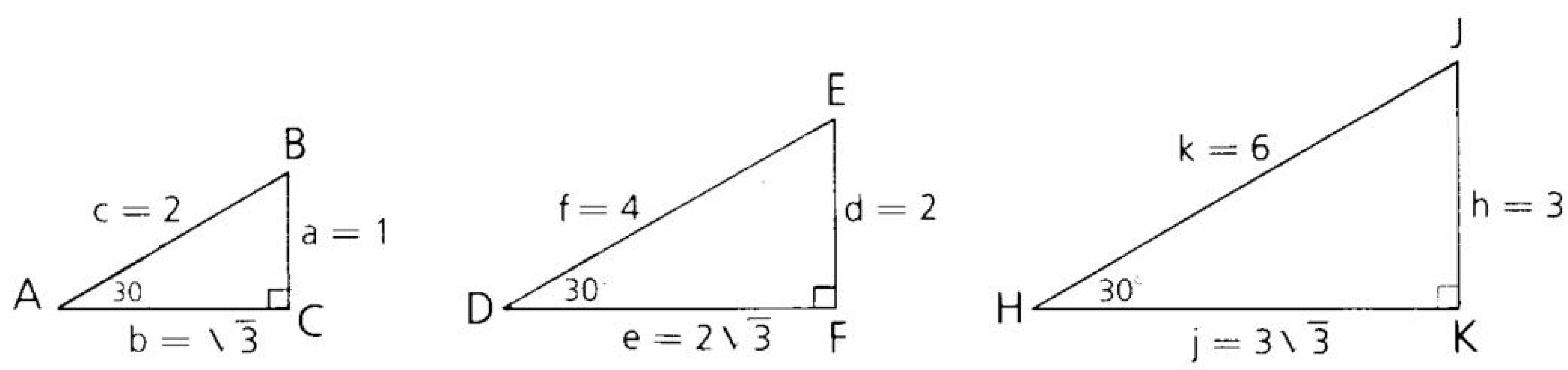
After studying this section, you will be able to

- Understand three basic trigonometric relationships

Part One: Introduction

This section presents the three basic trigonometric ratios ***sine***, ***cosine***, and ***tangent***. The concept of similar triangles and the Pythagorean Theorem can be used to develop the ***trigonometry of right triangles.***

Consider the following 30°-60°-90° triangles.



Compare the length of the leg opposite the 30° angle with the length of the hypotenuse in each triangle.

In $\triangle ABC$, $\frac{a}{c} = \frac{1}{2} = 0.5$. In $\triangle DEF$, $\frac{d}{f} = \frac{2}{4} = 0.5$. In $\triangle HJK$, $\frac{h}{k} = \frac{3}{6} = 0.5$.

If you think about similar triangles, you will see that in every 30°-60°-90° triangle,

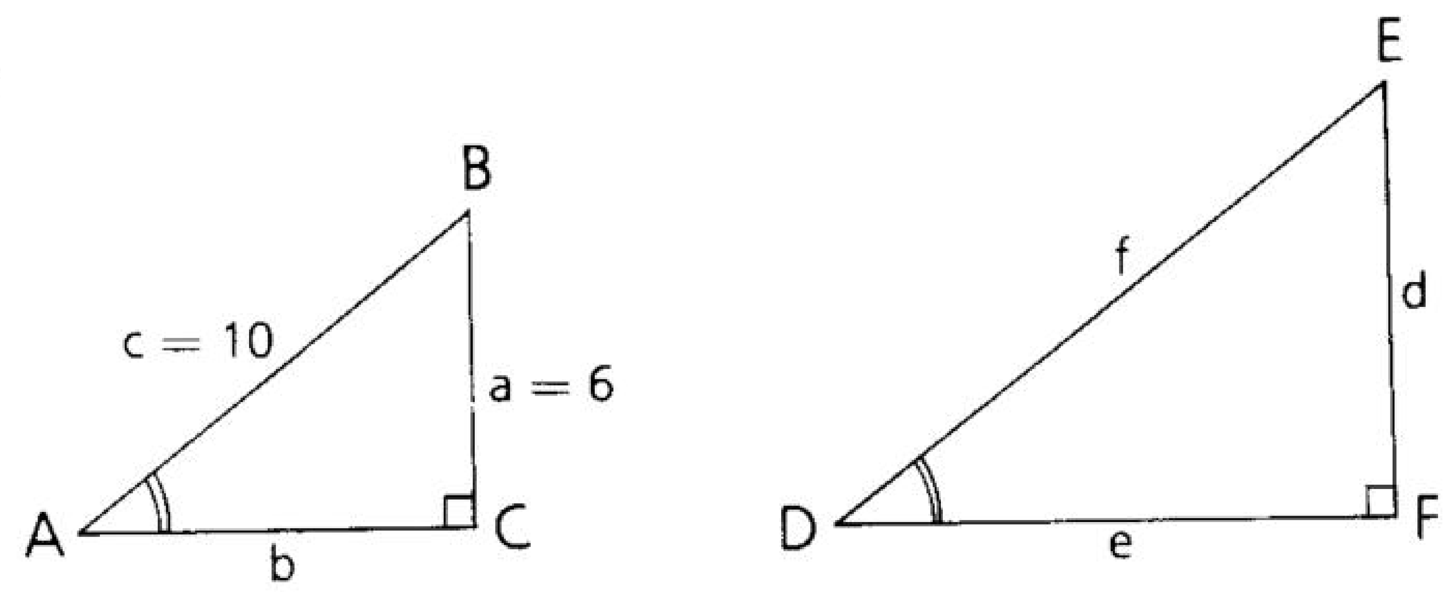
$$\frac{\text{leg opposite } 30° \angle}{\text{hypotenuse}} = \frac{1}{2}$$

For each triangle shown, verify that $\frac{\text{leg adjacent to } 30° \angle}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$.

For each triangle shown, find the ratio $\frac{\text{leg opposite } 30° \angle}{\text{leg adjacent to } 30° \angle}$.

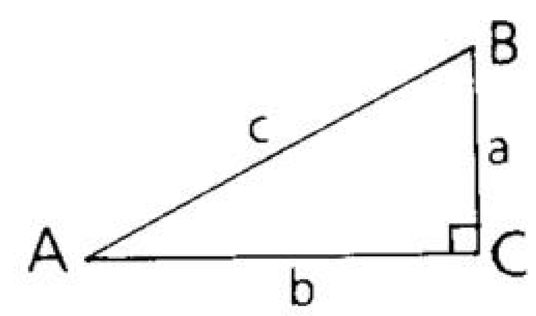
In $\triangle ABC$ and $\triangle DEF$,

$$\frac{a}{c} = \frac{d}{f} = \frac{6}{10} = \frac{3}{5}$$



Engineers and scientists have found it convenient to formalize these relationships by naming the ratios of sides. You should memorize these three basic ratios.

Definition **Three Trigonometric Ratios**



SOHCAHTOA

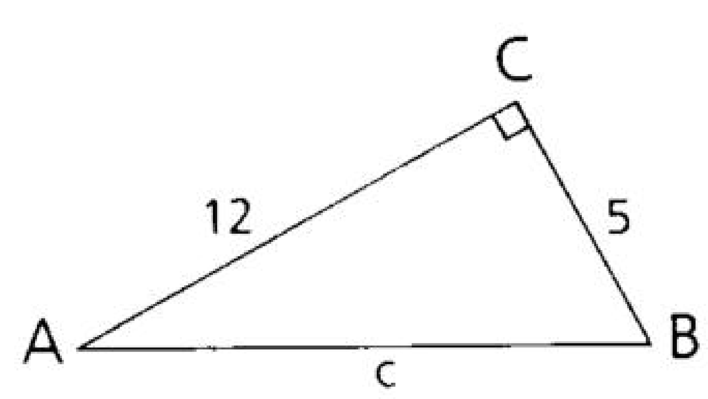
$$\textbf{\textit{sine}} \text{ of } \angle A = \sin \angle A = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

$$\textbf{\textit{cosine}} \text{ of } \angle A = \cos \angle A = \frac{\text{adjacent leg}}{\text{hypotenuse}}$$

$$\textbf{\textit{tangent}} \text{ of } \angle A = \tan \angle A = \frac{\text{opposite leg}}{\text{adjacent leg}}$$

Part Two: Sample Problems

Problem 1 *Find:* **a** *cos* $\angle A$
b *tan* $\angle B$



Solution By the Pythagorean Theorem, $c = 13$.

a $\cos \angle A = \dfrac{\text{leg adjacent to } \angle A}{\text{hypotenuse}} = \dfrac{12}{13}$

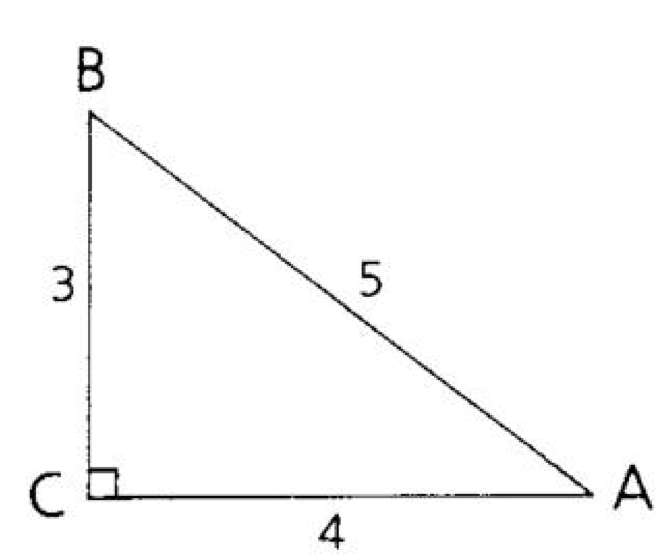
b $\tan \angle B = \dfrac{\text{leg opposite } \angle B}{\text{leg adjacent to } \angle B} = \dfrac{12}{5}$

Problem 2 *Find the three trigonometric ratios for* $\angle A$ *and* $\angle B$.

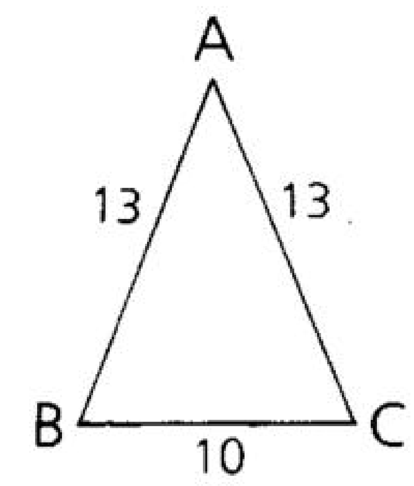
Solution $\sin \angle A = \dfrac{3}{5}$ $\sin \angle B = \dfrac{4}{5}$

$\cos \angle A = \dfrac{4}{5}$ $\cos \angle B = \dfrac{3}{5}$

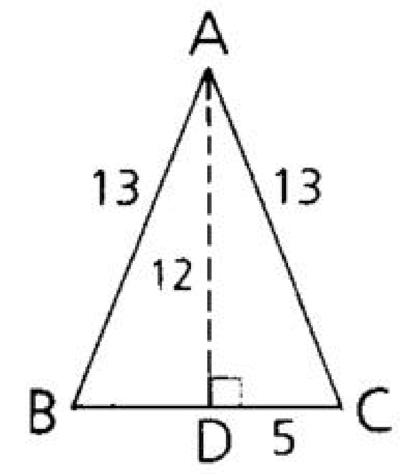
$\tan \angle A = \dfrac{3}{4}$ $\tan \angle B = \dfrac{4}{3}$



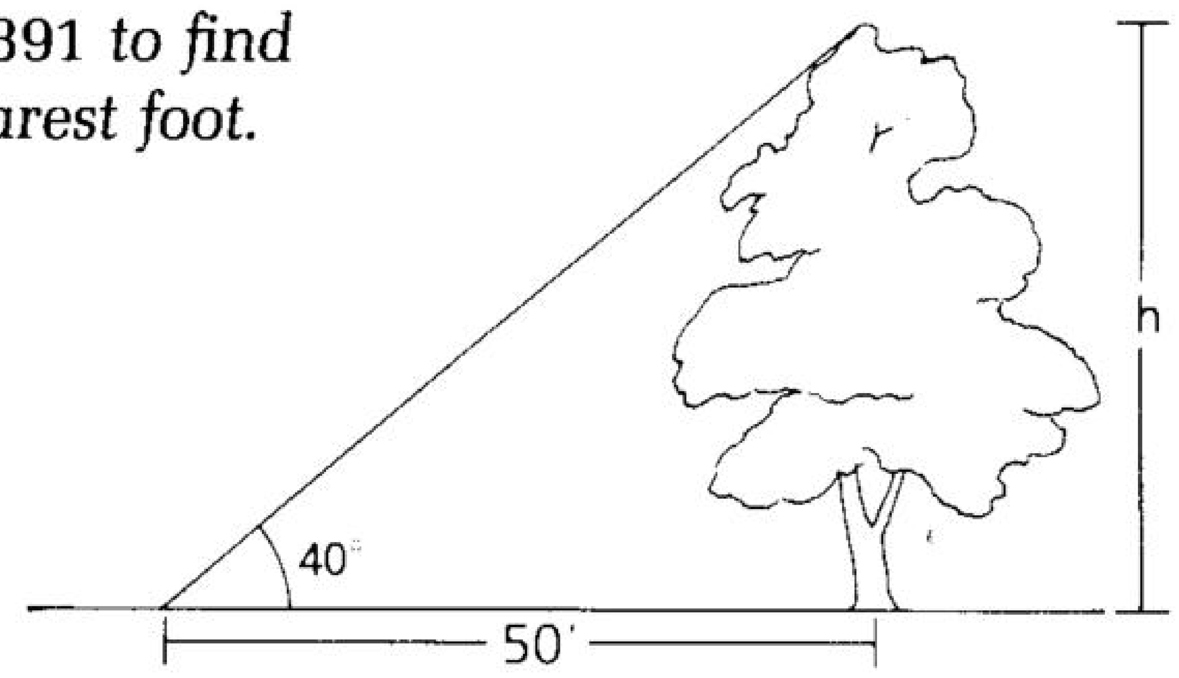
Problem 3 *$\triangle ABC$ is an isosceles triangle as marked. Find sin $\angle C$.*



Solution We must have a right triangle, so we draw the altitude to the base.
Thus, in $\triangle ADC$, $\sin \angle C = \frac{12}{13}$.



Problem 4 *Use the fact that tan 40° ≈ 0.8391 to find the height of the tree to the nearest foot.*



Solution

$$\tan 40° = \frac{h}{50}$$

$$0.8391 \approx \frac{h}{50}$$

$$h \approx 41.955$$

$$\approx 42 \text{ ft}$$

Part Three: Problem Sets

Problem Set A

1 Find each ratio.

a sin $\angle A$

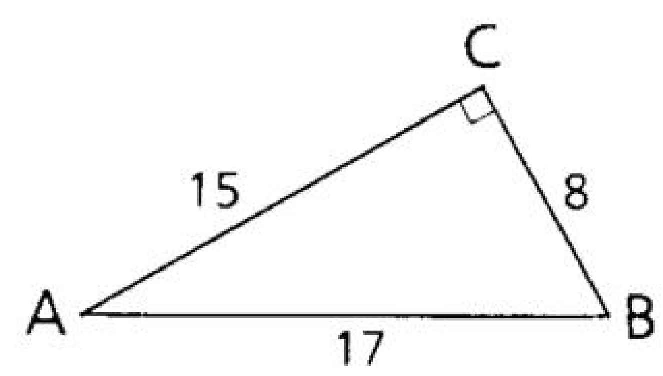
b cos $\angle A$

c tan $\angle A$

d sin $\angle B$

e cos $\angle B$

f tan $\angle B$



2 Find each ratio.

a sin 30°

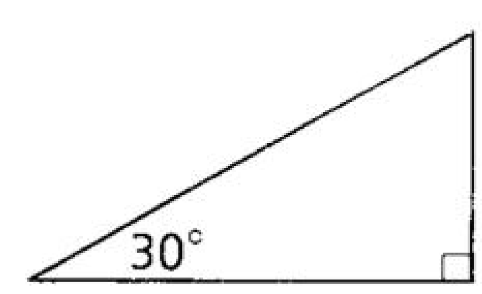
b cos 30°

c tan 30°

d sin 60°

e cos 60°

f tan 60°

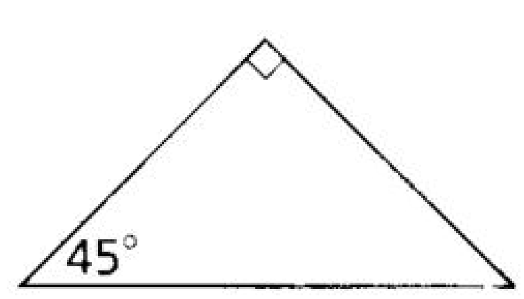


3 Find each ratio.

a sin 45°

b cos 45°

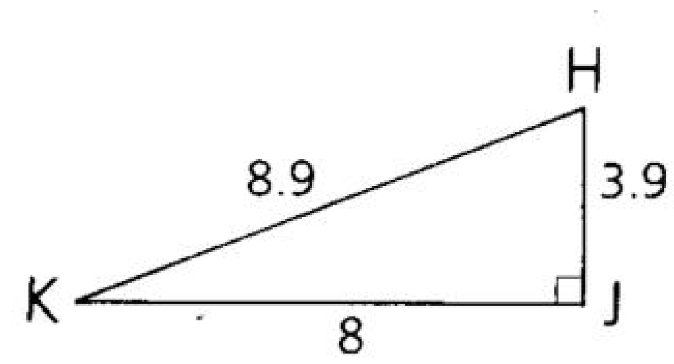
c tan 45°



4 Find each ratio.

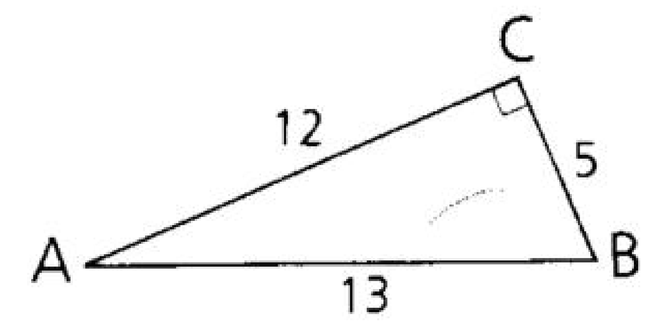
a cos $\angle H$

b tan $\angle K$



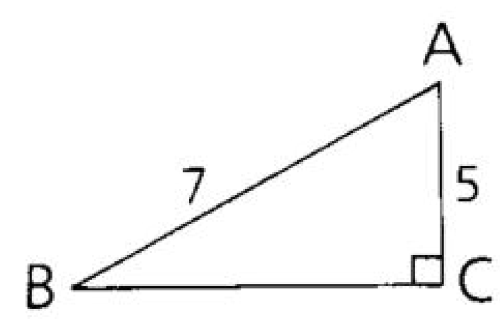
5 If tan $\angle M = \frac{3}{4}$, find cos $\angle M$. (Hint: Start by drawing the triangle.)

6 Using the figure as marked, name each missing angle.



a $\frac{5}{12}$ = tan $\angle$ __?__ **b** $\frac{12}{13}$ = cos $\angle$ __?__ **c** $\frac{5}{13}$ = sin $\angle$ __?__

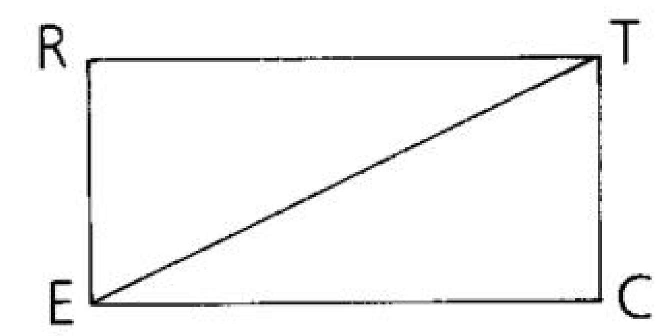
7 Find each quantity.



a BC **b** sin $\angle A$ **c** tan $\angle B$

8 Given: RECT is a rectangle.
ET = 26, RT = 24

Find: **a** sin $\angle RET$ **b** cos $\angle RET$



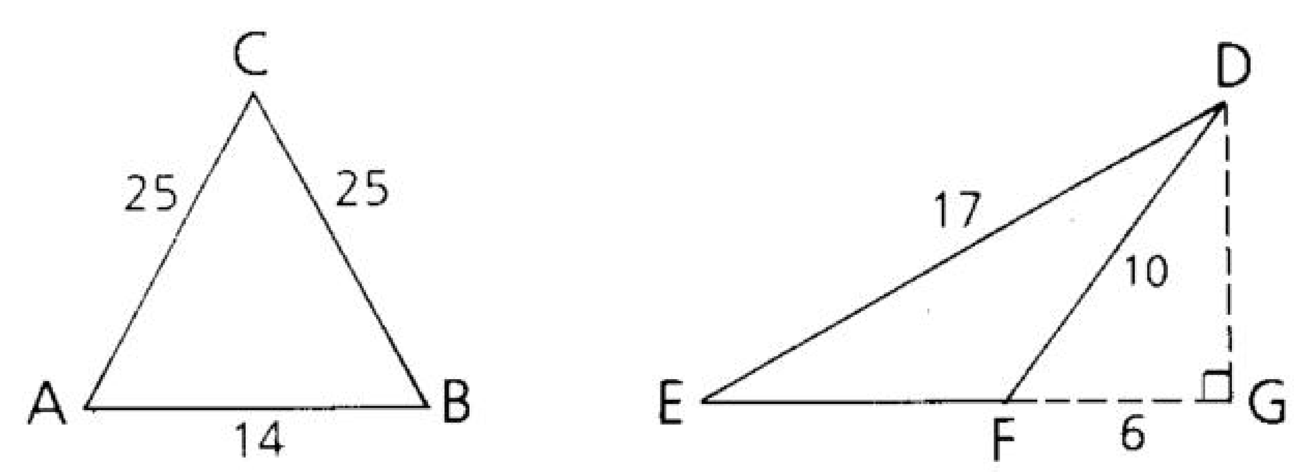
Problem Set B

9 Using the given figures, find

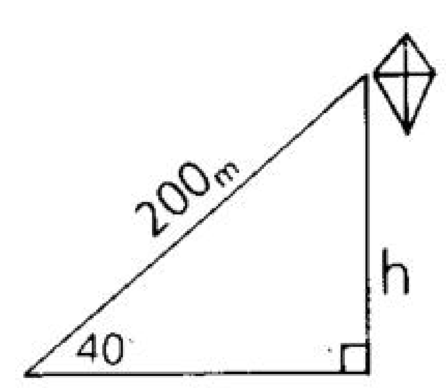
a cos $\angle A$

b sin $\angle E$

c sin $\angle DFG$



10 Use the fact that sin 40° ≈ 0.6428 to find the height of the kite to the nearest meter.

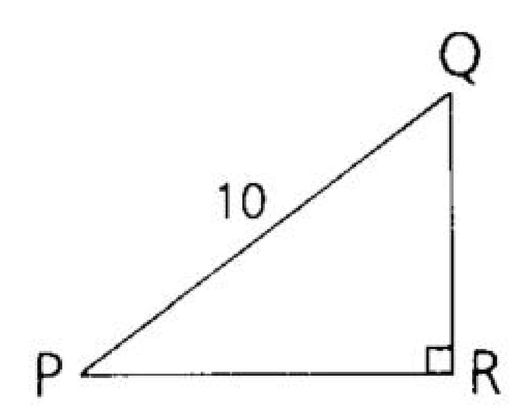


11 **a** If tan $\angle A$ = 1, find m$\angle A$.

b If sin $\angle P$ = 0.5, find m$\angle P$.

12 Given: sin $\angle P = \frac{3}{5}$, PQ = 10

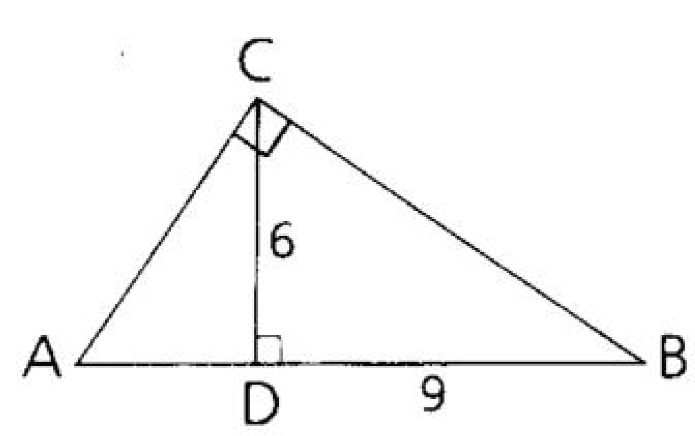
Find: cos $\angle P$



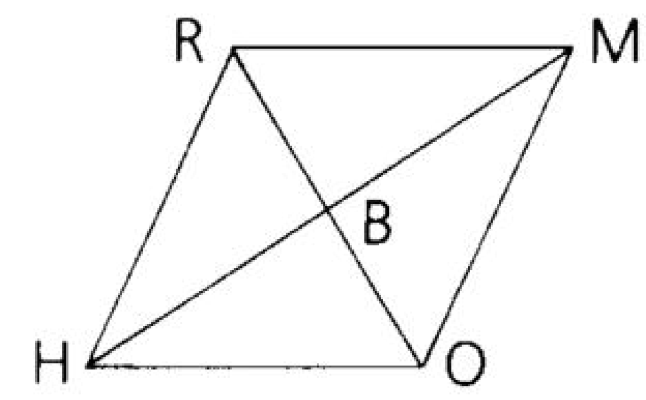
13 Using the figure, find

a tan $\angle ACD$

b sin $\angle A$



Problem Set B, *continued*



14 Given: RHOM is a rhombus.
RO = 18, HM = 24

Find: **a** cos $\angle$BRM **b** tan $\angle$BHO

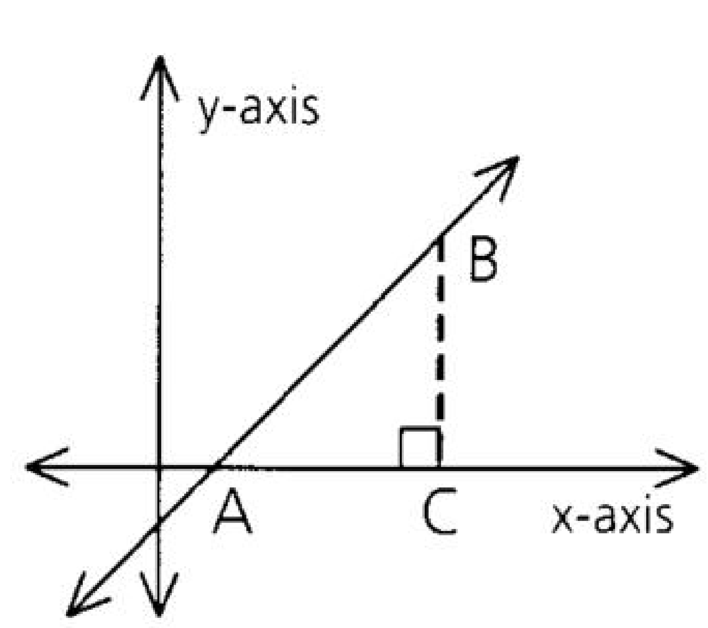
15 Given a trapezoid with sides 5, 10, 17, and 10, find the sine of one of the acute angles.

16 Given $\triangle$ABC with $\angle$C = 90°, indicate whether each statement is true Always (A), Sometimes (S), or Never (N).

a $\sin \angle A = \cos \angle B$ **b** $\sin \angle A = \tan \angle A$ **c** $\sin \angle A = \cos \angle A$

17 If $\triangle$EQU is equilateral and $\triangle$RAT is a right triangle with RA = 2, RT = 1, and $\angle$T = 90°, show that $\sin \angle E = \cos \angle A$.

18 If the slope of $\overleftrightarrow{AB}$ is $\frac{5}{8}$, find the tangent of $\angle$BAC.



PROOF

Problem Set C

19 Use the definitions of the trigonometric ratios to verify the following relationships, given $\triangle$ABC in which $\angle$C = 90°.

a $(\sin \angle A)^2 + (\cos \angle A)^2 = 1$

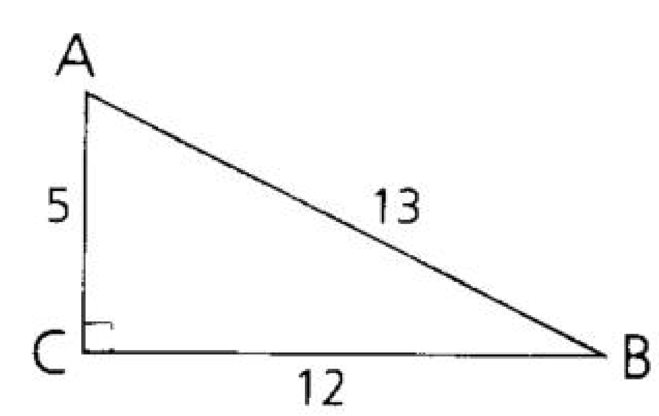
b $\dfrac{a}{\sin \angle A} = \dfrac{b}{\sin \angle B}$

c $\dfrac{\sin \angle A}{\cos \angle A} = \tan \angle A$

d $\sin \angle A = \cos (90° - \angle A)$

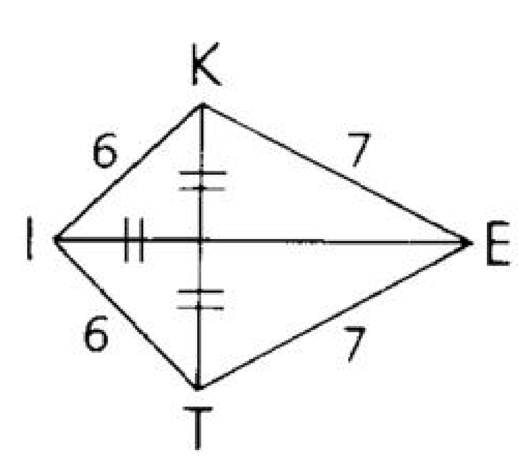
20 Rhombus PQRS has a perimeter of 60 and one diagonal of 15. Find the two possible values of sin $\angle$PQS.

21 Two sides of the triangle shown are picked at random to form a ratio. What is the probability that the ratio is the tangent of $\angle$A?



22 Given: KITE is a kite with sides as marked.

Find: tan $\angle$KEI



9.10 TRIGONOMETRIC RATIOS

Objective

After studying this section, you will be able to

- Use trigonometric ratios to solve right triangles

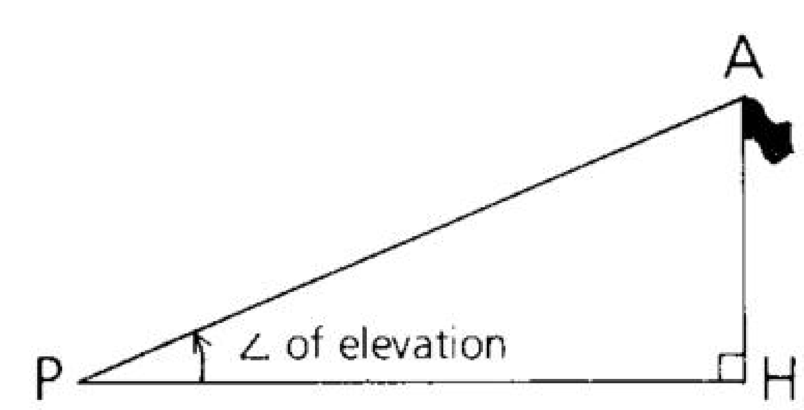
Part One: Introduction

Trigonometry is used to solve triangles other than 30°-60°-90° and 45°-45°-90° triangles. The Table of Trigonometric Ratios on the next page shows four-place decimal approximations of the ratios for other angles—for instance, sin 23° ≈ 0.3907, and the angle whose tangent is 1.5399 is approximately 57°.

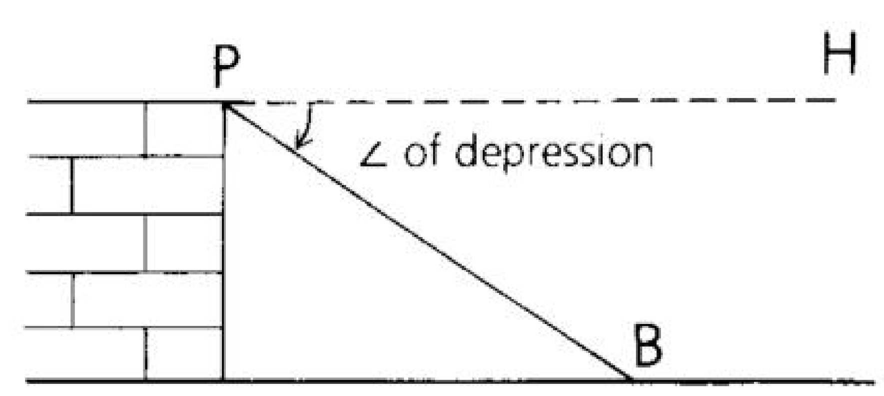
Unless your teacher directs otherwise, we suggest you use a scientific calculator rather than the table to find trigonometric ratios.

For some applications of trigonometry, you need to know the meanings of ***angle of elevation*** and ***angle of depression***.

If an observer at a point P looks upward toward an object at A, the angle the line of sight $\overrightarrow{PA}$ makes with the horizontal $\overrightarrow{PH}$ is called the ***angle of elevation***.



If an observer at a point P looks downward toward an object at B, the angle the line of sight $\overrightarrow{PB}$ makes with the horizontal $\overrightarrow{PH}$ is called the ***angle of depression***.



Note Do not forget that an angle of elevation or depression is an angle between a line of sight and the *horizontal*. Do not use the vertical.

Table of Trigonometric Ratios

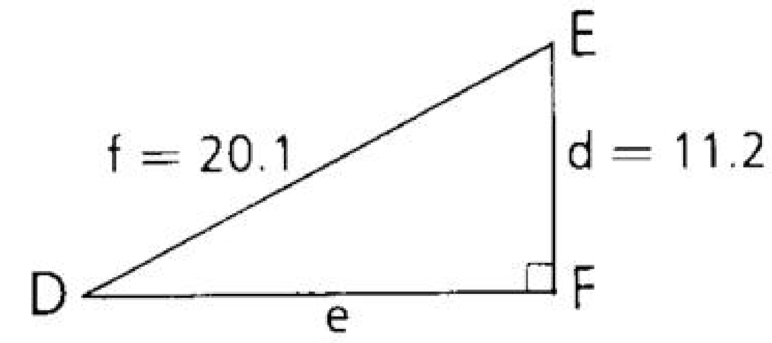
∠A	sin ∠A	cos ∠A	tan ∠A	∠A	sin ∠A	cos ∠A	tan ∠A
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

Part Two: Sample Problems

Problem 1 Given: *Right △DEF as shown*

Find: **a** m∠D *to the nearest degree*

b *e to the nearest tenth*



Solution

a $\sin \angle D = \dfrac{11.2}{20.1}$

$\sin \angle D \approx 0.5572$

The number nearest to 0.5572 in the sine column of the table is sin 34°, so ∠D ≈ 34°.

b We use the result from part **a**.

$\cos 34° \approx \dfrac{e}{20.1}$

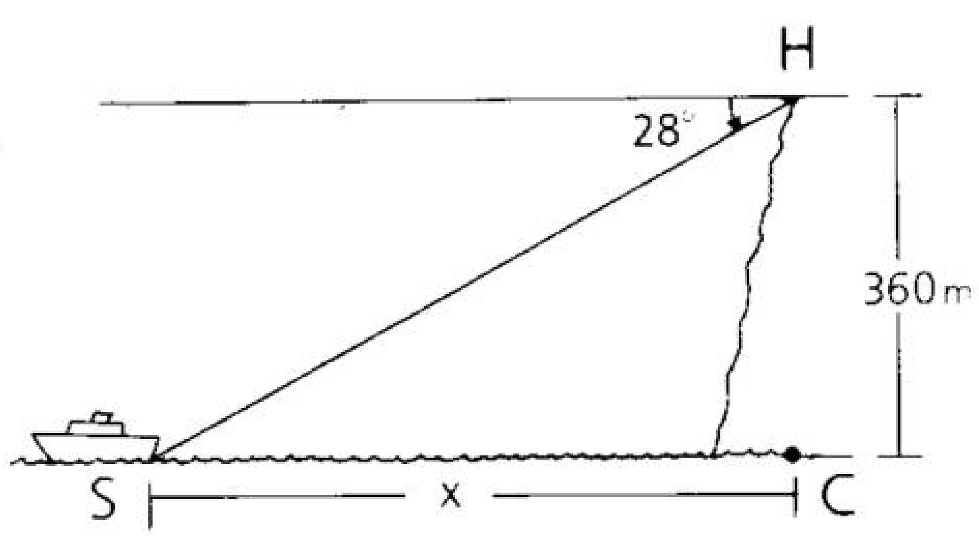
$0.8290 \approx \dfrac{e}{20.1}$

$16.7 \approx e$

Problem 2 *To an observer on a cliff 360 m above sea level, the angle of depression of a ship is 28°. What is the horizontal distance between the ship and the observer?*

Solution Start by drawing a diagram.

By ∥ lines ⇒ alt. int. ∠s ≅, ∠CSH = 28°.



Thus, $\tan 28° = \dfrac{360}{x}$

$0.5317 \approx \dfrac{360}{x}$

$x \approx 677$

The horizontal distance is about 677 m.

Part Three: Problem Sets

Problem Set A

1 Find each of the following in the Table of Trigonometric Ratios.

a sin 21° **b** tan 52° **c** cos 5° **d** tan 45° **e** sin 60°

2 Using the table, find m∠A in each case.

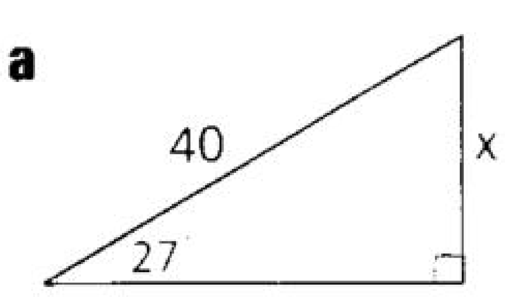
a sin ∠A = 0.4067 **b** tan ∠A = 3.4874 **c** cos ∠A = .7071

3 Without using the table, find m∠A in each case.

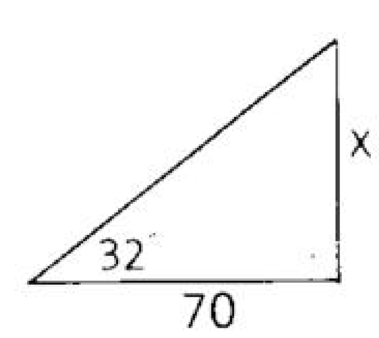
a tan ∠A = 1 **b** $\sin \angle A = \dfrac{1}{2}$ **c** $\sin \angle A = \dfrac{\sqrt{3}}{2}$

4 In each case, find x to the nearest integer.

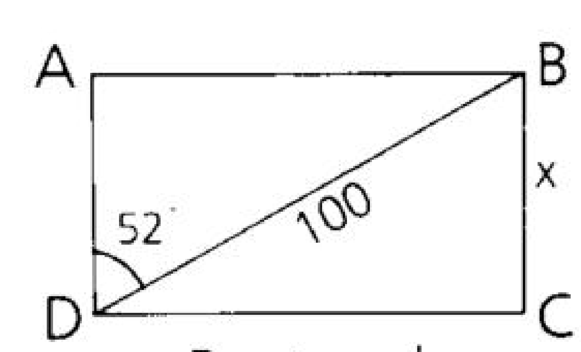
a



b

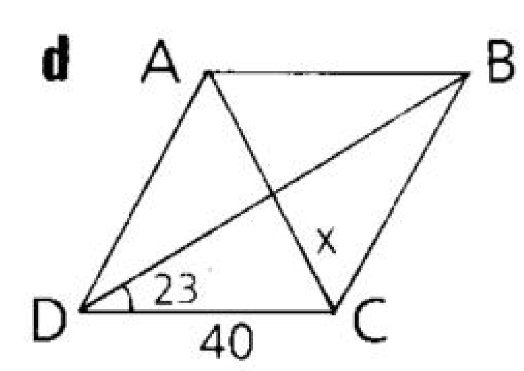


c



Rectangle

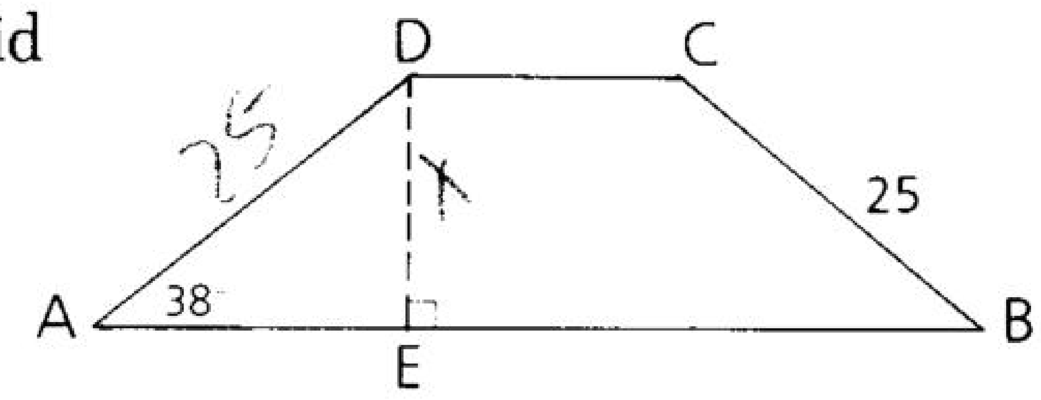
d



Rhombus

Problem Set A, *continued*

5 Find the height of isosceles trapezoid ABCD.



Problem Set B

6 Solve each equation for x to the nearest integer.

a $\sin 25° = \frac{x}{40}$ **b** $\cos 73° = \frac{35}{x}$ **c** $\sin x° = \frac{29}{30}$

7 A department-store escalator is 80 ft long. If it rises 32 ft vertically, find the angle it makes with the floor.

8 Given the regular pentagon shown, with center at O and EN = 12 cm,

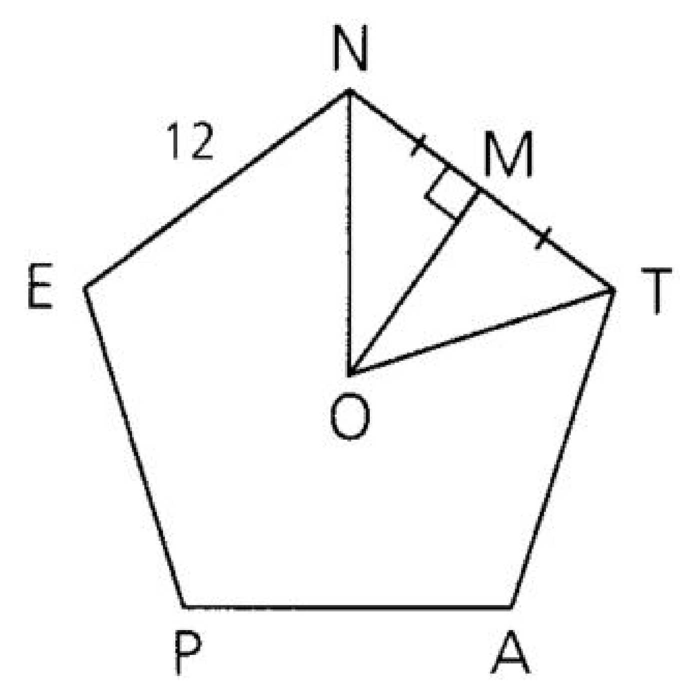
a Find m∠E

b Find m∠NOM

c Find OM to the nearest hundredth

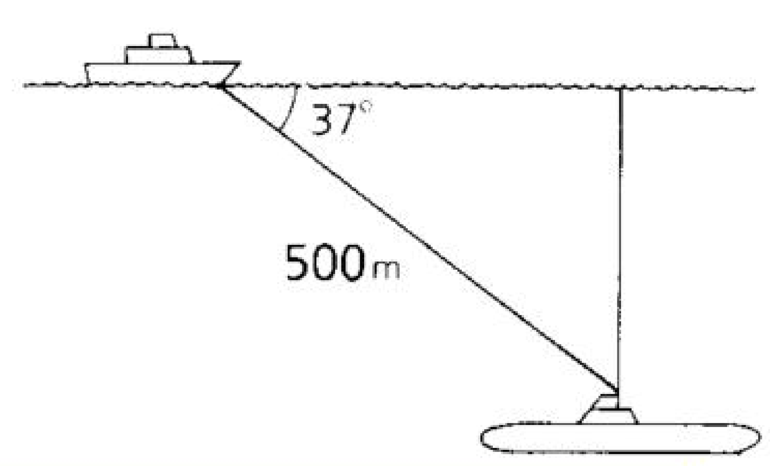
d Find the area of △NOT to the nearest hundredth

e Explain how you could find the area of the pentagon



9 Find, to the nearest degree, the angles of a (3, 4, 5) triangle.

10 A sonar operator on a cruiser detects a submarine at a distance of 500 m and an angle of depression of 37°. How deep is the sub?



11 The legs of an isosceles triangle are each 18. The base is 14.

a Find the base angles to the nearest degree.

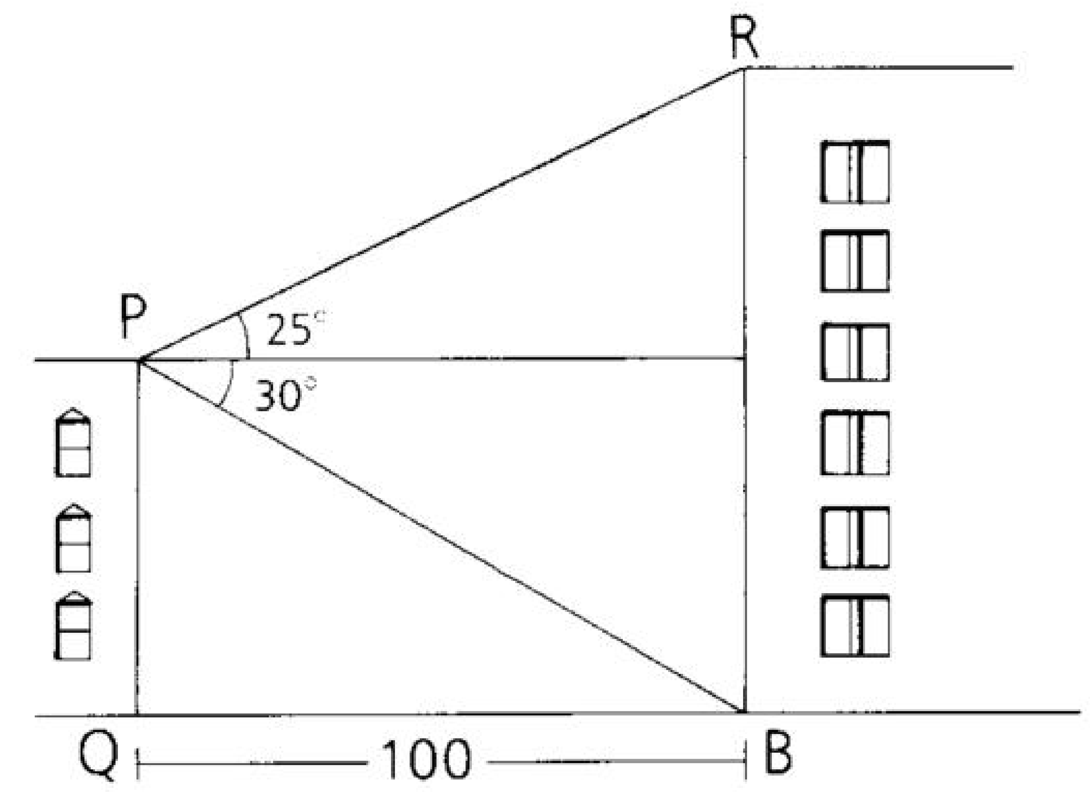
b Find the exact length of the altitude to the base.

12 One diagonal of a rhombus makes an angle of 27° with a side of the rhombus. If each side of the rhombus has a length of 6.2 in., find the length of each diagonal to the nearest tenth of an inch.

13 Find the perimeter of trapezoid ABCD, in which $\overline{CD} \parallel \overline{AB}$, $\cos \angle A = \frac{1}{2}$, and AD = DC = CB = 2.

14 Find the length of the apothem of a regular pentagon that has a perimeter of 50 cm.

15 Two buildings are 100 dm apart across a street. A sunbather at point P finds the angle of elevation of the roof of the taller building to be 25° and the angle of depression of its base to be 30°. Find the height of the taller building to the nearest decimeter.

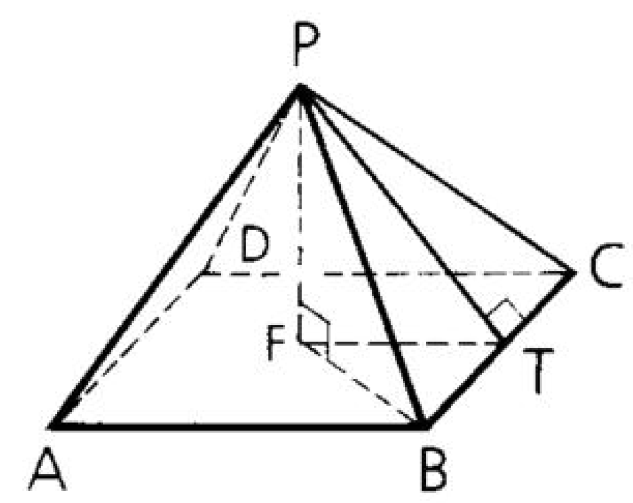


Problem Set C

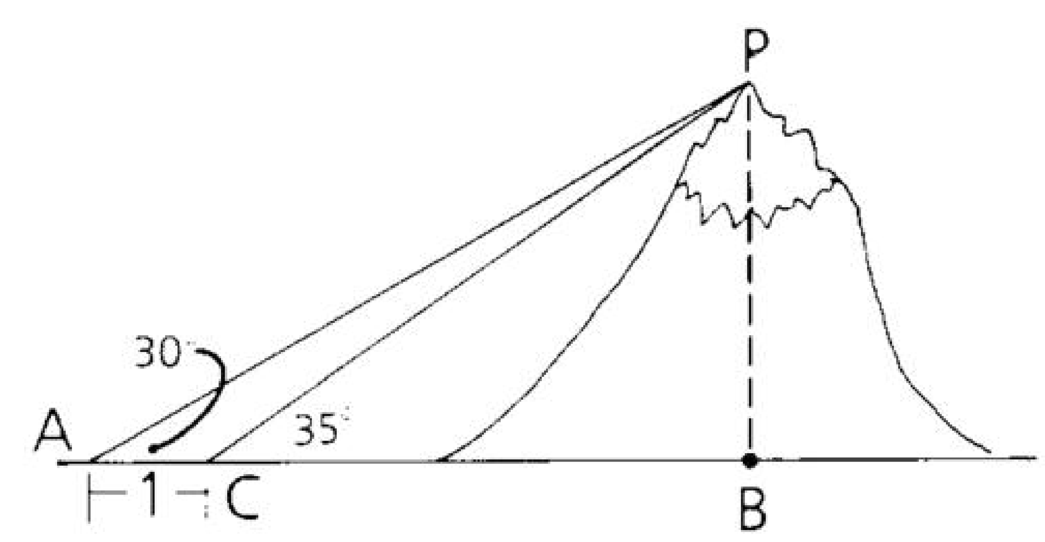
16 An observer on a cliff 1000 dm above sea level sights two ships due east. The angles of depression of the ships are 47° and 32°. Find, to the nearest decimeter, the distance between the ships.

17 Each side of the base of a regular square pyramid is 20 and the altitude is 35.

Find: **a** PT **b** BP **c** ∠PTF **d** ∠PBF

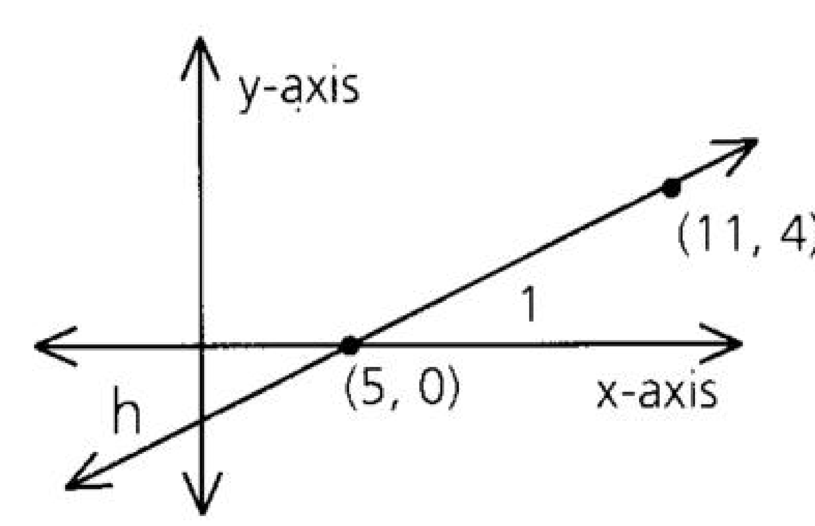


18 Find the height, PB, of a mountain whose base and peak are inaccessible. At point A the angle of elevation of the peak is 30°. One kilometer closer to the mountain, at point C, the angle of elevation is 35°.



19 **a** Find the slope of line h.

b Find m∠1 to the nearest integer.



Problem Set D

20 Prove that $c^2 = a^2 + b^2 - 2ab(\cos \angle C)$ is true for any acute △ABC. (This formula is called the Law of Cosines.)

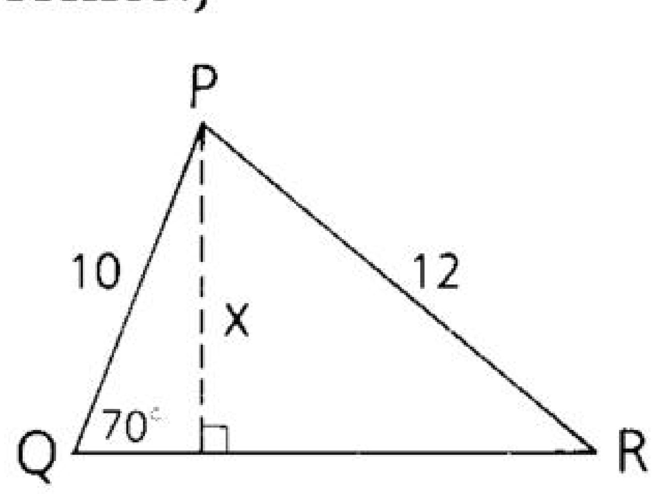
21 Given: Diagram as shown

a Find ∠R to the nearest degree.

b Find QR to the nearest integer.

c Show that $\frac{PR}{\sin \angle Q} = \frac{PQ}{\sin \angle R}$.

d Generalize the result of part **c** for the sides and angles of any acute triangle. (The resulting formula is the Law of Sines.)



9 CHAPTER SUMMARY

CONCEPTS AND PROCEDURES

After studying this chapter, you should be able to

- Simplify radical expressions and solve quadratic equations (9.1)
- Begin solving problems involving circles (9.2)
- Identify the relationships between the parts of a right triangle when an altitude is drawn to the hypotenuse (9.3)
- Use the Pythagorean Theorem and its converse (9.4)
- Use the distance formula to compute lengths of segments in the coordinate plane (9.5)
- Recognize groups of whole numbers known as Pythagorean triples (9.6)
- Apply the Principle of the Reduced Triangle (9.6)
- Identify the ratio of side lengths in a 30°-60°-90° triangle (9.7)
- Identify the ratio of side lengths in a 45°-45°-90° triangle (9.7)
- Apply the Pythagorean Theorem to solid figures (9.8)
- Understand three basic trigonometric relationships (9.9)
- Use trigonometric ratios to solve right triangles (9.10)

VOCABULARY

altitude (9.8)
angle of depression (9.10)
angle of elevation (9.10)
base (9.8)
cosine (9.9)
cube (9.8)
diagonal (9.8)
distance formula (9.5)
edge (9.8)
face (9.8)
Pythagorean triple (9.6)
rectangular solid (9.8)
regular square pyramid (9.8)
sine (9.9)
slant height (9.8)
tangent (9.9)
trigonometry of right triangles (9.9)
vertex (9.8)